Lower-Dimensional Complex Manifolds in Several Complex Variables

by

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CHAPTER 1

Introduction and background

1. Philosophy

Given an n-dimensional manifold X, there are several ways to relate the structure of X to manifolds W^m of lower dimension m. For instance, one can consider the m-dimensional submanifolds W^m of X, for example, and ask in which ways the topology and structure of W^m are inherited from that of X. Alternatively, one can change the topology and structure of X by "gluing" more manifolds W^m to the boundary ∂X.

When one works in Several Complex Variables, the manifolds X of interest are those manifolds which carry with them a complex (or related) structure of some nature: complex analytic structure, symplectic structure, a Cauchy-Riemann (C-R) stucture, q-convex structure, or Stein structure. If X is a manifold embedded in $\mathbb{C}^k$ for some k, it can automatically inherit a complex structure from the ambient space. In the case of Stein manifolds X, for example, it is always possible to embed X in an ambient space, but even here it is still normally more useful to think of the complex structure as being intrinsic.

In this thesis, I consider two problems relating lower-dimensional manifolds to higher-dimensional manifolds with complex structure:

Question. Given a complex analytic manifold X, what conditions do we need on X so that for every point $p \in X$ there is a Stein manifold M without boundary such that $p \in M \subset X$?

The question is equivalent to asking when, for every point p in X, there is a way of finding p in a complex line within X. I have shown that it is possible to do so for a large subclass of q-convex manifolds called q-complete manifolds. For these manifolds, we can find p in a proper holomorphic image $F(\Delta)$ of the unit disc in X.

Question. If X is a strictly pseudoconvex domain (and hence *a forteriori* a Stein manifold), how can we attach lower-dimensional manifolds Σ to it in such a way that the complex structure is preserved for the union of the manifolds X and Σ?

Assume Σ is "totally real" - that is, has no complex structure of its own. If Σ also satisfies certain tangency conditions on the set where where Σ meets the boundary ∂X of X, then this thesis provides a proof that Σ can inherit complex structure from X. We say that Σ is a "handle" which has been "glued" to X.

Chapter 2 addresses the first of these two questions, and Chapter 3 addresses the second. The treatment in Chapter 2 extends arguments due to Forstnerič, Globevnik and Stensønes to the case of q-complete manifolds. The argument draws on many aspects of complex analysis, and uses two lemmas that extend the clever (and complicated) main lemmas from [**7**] and [**15**].

In contrast, the treatment of handles in Chapter 3 is entirely elementary. First, the "standard" handle (comprised of the unit ball and a flat plane in $\mathbb{C}^2$) is shown to have a neighborhood basis of strictly pseudoconvex sets. This initial demonstration depends only on symmetry arguments and some elementary analysis. The result is extended to higher dimensions and more general strictly pseudoconvex domains by viewing them as locally similar to the canonical case of the standard handle.

2. Previous research

2.1. Proper Embeddings. The existence of a proper holomorphic map F from a domain D to another domain or manifold Ω has long been a matter of interest to those who study Several Complex Variables. (For an excellent review paper, see [**6**]). The fact that F is holomorphic means that the complex structure of $F(D)$ (and therefore D) is inherited from that of Ω. Properness of F ensures that no information on D is lost, and that we can therefore consider the complex structure of D itself as being inherited from that of Ω. It also implies that $F(D)$ is closed in Ω.

In a paper published in 1992, Forstneričand Globevnik [**7**] showed the following: For any pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with $\mathbb{C}^2$ boundary and any prescribed point $p \in \Omega$, it is possible to find a proper holomorphic map F from the unit disc $\Delta \subset \mathbb{C}$ into Ω such that $F(0) = p$. Furthermore, if a direction λ is given, F can be found such that $F'(0) = b\lambda$ for some $b > 0$. Also in 1992, Berit Stensønes [**15**] extended this result to the case of any Stein manifold of dimension ≥ 2, and in this thesis, it is extended to the class of q-complete manifolds.

2.2. Handles. In 1977, Fornæss and Stout [**4**] showed that, for any pseudoconvex domain Ω and strictly pseudoconvex point $p \in \partial\Omega$, one can attach a one-dimensional "handle" to Ω in the sense that there exists a Stein neighborhood basis for the set $\overline{\Omega} \cup L$, where L is the line through p perpindicular to $\partial\Omega$.

In 1990, Yakov Eliashberg [**3**] extended this result considerably in a paper wherein he showed that a Stein manifold X of complex dimension n with an exhaustion function that has critical points of index $\leq n$ admits a filtration

$$X_1 \subset\subset X_2 \subset\subset \ldots \subset\subset X_n$$

by Stein manifolds X_i where each X_i arises from X_{i-1} by attaching handles of dimension no greater than i. His paper contained, as a crucial lemma, the idea that one can attach higher dimensional handles to compact complex manifolds with pseudoconvex boundary, and hence in particular to pseudoconvex domains. The proof is difficult, however, and is aimed only at attaching handles up to topological equivalence, so this thesis provides a more elementary, concrete demonstration for strictly pseudoconvex domains.

3. Basic definitions

The following concepts will be used throughout this thesis. Where possible, the notation of these definitions will also be preserved.

DEFINITION 3.1. A map f is *proper* if the preimage of any compact set in the range of f is compact in the domain of f. Intuitively, this is means that the boundary of the domain of f must be mapped into the boundary of the range.

DEFINITION 3.2. $f(z) = (f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n))$ is a *holomorphic* map if each of the coordinate functions f_j are holomorphic, i.e. if each f_j is analytic in each variable seperately.

DEFINITION 3.3. A *defining function* for a domain Ω is a continuous function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $\Omega = \{z \ : \ \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. An *exhaustion function* is a continuous function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that if we set $\Omega_c = \{z \ : \ \rho(z) < c\}$, then $\Omega = \bigcup\limits_{m \in \mathbb{N}} \Omega_m$, and the level sets satisfy $\Omega_c \subset\subset \Omega_{c+a}$ for all $a \in \mathbb{R}$.

DEFINITION 3.4. A vector λ in $\mathbb{C}^n \setminus \{0\}$ is *complex tangent* to the boundary of Ω at p if it satisfies the equation $\sum \frac{\partial \rho}{\partial z_i}\Big|_p \lambda_i = 0$ where ρ is a $\mathcal{C}^2$ defining function for Ω at p.

DEFINITION 3.5. A $\mathcal{C}^2$ function ρ is *plurisubharmonic* at z in $\mathbb{C}^n$ if its *Levi form* satisfies $\mathcal{L}_\rho(z, \lambda) = \sum\limits_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \lambda_j \overline{\lambda}_k \geq 0$ for all complex directions $\lambda \in \mathbb{C}^n \setminus \{0\}$ – that is, if $\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)$ is a positive semidefinite matrix.

DEFINITION 3.6. A $\mathcal{C}^2$ function ρ is *q-subharmonic* at z in $\mathbb{C}^n$ if $\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)$ has at least q distinct eigenvectors with positive eigenvalues.

DEFINITION 3.7. The *Levi polynomial* L_p for ρ at p is the second degree polynomial $\sum\limits_{j=1}^{n} \frac{\partial \rho}{\partial z_j}\Big|_p (z_j - p_j) + \frac{1}{2} \sum\limits_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}\Big|_p (z_j - p_j)(z_k - p_k)$. The *Levi support surface* for ρ at p is the surface of $2n-1$ real dimensions defined by the equation $L_p = 0$. Note that if ρ is plurisubharmonic then the point p is a local minimum for ρ on the Levi support surface.

DEFINITION 3.8. A domain Ω is *pseudoconvex* if there exists a plurisubharmonic *exhaustion function* ρ for Ω. That is, ρ is a function such that the level sets $\Omega_c = \{z : \rho(z) < c\}$ are all relatively compact in Ω and $\rho(z)$ is plurisubharmonic for all $p \in \partial\Omega$. We say Ω is *pseudoconvex in the sense of Levi* if there exists a $\mathcal{C}^2$ defining function ρ such that $\nabla\rho|_{\partial\Omega} \neq 0$ and for all $p \in \partial\Omega$ and complex tangent vectors λ to $\partial\Omega$ at p, the Levi form $\sum\limits_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \lambda_j \overline{\lambda}_k \geq 0$. Note that pseudoconvexity in the sense of Levi automatically implies pseudoconvexity, but that the converse is true only if $\partial\Omega$ is $\mathcal{C}^2$ smooth.

DEFINITION 3.9. A domain Ω is *q-convex* if there exists an exhaustion function ρ for Ω which is q-subharmonic on all but a compact subset of points V_Ω in Ω. A manifold X is *q-convex* if it admits an exhaustion function which is q-subharmonic on all but a compact subset of points V_X in X.

DEFINITION 3.10. If the inequalities in Definitions 3.5, 3.6, 3.8 and 3.9 are strict, then they define (respectively) the terms: strictly plurisubharmonic, strictly q-subharmonic, strictly pseudoconvex, and strictly q-convex.

DEFINITION 3.11. A *Stein manifold* M is a complex analytic manifold with a strictly plurisubharmonic exhaustion function ϕ such that

$$M_c = \{z \in M : \phi(z) < c\} \subset\subset M$$

for all c, and the set of functions $\{f : f \text{ is holomorphic on } M\}$ seperates points.

CHAPTER 2

Disc Embeddings

1. Overview

Our goal here is to prove that the broad class of complex analytic manifolds known as q-complete manifolds have the property that a proper analytic disk can be found through any prescribed point inside them.

DEFINITION 1.1. Let Ω be a σ-finite complex analytic manifold which admits a smooth exhaustion function ρ. Assume that for all z in Ω, ρ has a Levi form $\mathcal{L}$ with q strictly positive eigenvectors, all perpindicular to $\nabla\rho$ when $\nabla\rho \neq 0$ or $q+1$ strictly positive eigenvectors when $\nabla\rho = 0$. If, in addition, the holomorphic functions on Ω seperate points, then we say that Ω is *q-complete*.

EXAMPLE 1. Assume Ω is a smooth 2-convex manifold on which the holomorphic functions separate points, and the compact set V_Ω on which the exhaustion function fails to be q-subharmonic is empty. Then Ω is 1-complete.

The key for a holomorphic map from Δ into Ω (also known as an analytic disk) to be proper is for it to take the boundary of the disk to the boundary of Ω. Hence the idea of this proof is to begin with a small analytic disk through p with the correct derivative, and then to use it as the first in a sequence of analytic disks whose boundaries are pulled out closer and closer to the boundary of Ω. We proceed by means of a series of lemmas, each of which begins with an old analytic disk and certain conditions and provides a new analytic disk whose boundary is closer to that of Ω. Finally, we take a limit of these disks, which will be proper.

Structurally, Lemmas 2.9 and 2.10 provide the basic tool for "pulling" an old analytic disk out to the new one. Lemma 2.11 uses this tool when critical points of ρ are not nearby, while Lemmas 2.13 through 2.17 set up Lemma 2.18 to do so near a critical point.

Our work in Lemma 2.18 requires the following lemma, which adapts the proof of Theorem 1.4.15 in [**9**] to the case of q-subharmonic functions. It will allow us to write down a standard form for ρ in a neighborhood of any point – in particular near a critical point of ρ.

LEMMA 1.2. *If ρ is a q-subharmonic $\mathcal{C}^2$ function in a neighborhood of* 0 *in* $\mathbb{C}^n$, *then there exists a complex linear isometry* $t : \mathbb{C}^n \to \mathbb{C}^n$ *and* $\{\lambda_j\}_{j=1}^q$ *where* $\lambda_j \geq 0$ *such that (with* $z_j = x_j + ix_{q+j}$ *when* $j \leq q$*),*

$$\left(\frac{\partial^2 \rho \circ t(0)}{\partial x_j \partial x_k}\right)_{j,k=1}^{2q} = \begin{pmatrix} I_q + \Lambda & 0 \\ 0 & I_q - \Lambda \end{pmatrix} \tag{1.1}$$

where I_q *is the* $q \times q$ *identity matrix and* Λ *is the* $q \times q$ *diagonal matrix with entries* λ_j.

PROOF. Since the Levi matrix $L = \left(\frac{\partial^2 \rho(0)}{\partial z_j \partial \overline{z}_k}\right)_{j,k=1}^n$ has q positive eigenvectors, there exists an invertible complex matrix V such that

$$\begin{aligned} V^t L \overline{V} &= (\delta_{jk})_{j,k=1}^q + (\pm\delta_{jk})_{j,k=q+1}^n \\ &= I_{q+\ell,m} \end{aligned}$$

where $I_{q+\ell,m}$ is a $(q+\ell, m)$ *signature matrix* of the form

$$\begin{pmatrix} I_{q+\ell} & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\ell \geq 0$ and $m \leq n - q - \ell$. Let v be the complex linear isomorphism defined by V.

Without loss of generality, $\rho(0) = 0$ and $d\rho(0) = 0$, so that the Levi polynomial for ρ at 0 is simply $\sum \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} z_j z_k$ and the Taylor expansion for ρ at 0 is given by

$$\begin{aligned} \rho(z) &= |z_1|^2 + \cdots + |z_q|^2 \pm |z_{q+1}|^2 \pm |z_n|^2 + \Re\left[\sum_{j,k=1}^n \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} z_j z_k\right] + o\left(|z|^2\right) \\ &= |z_1|^2 + \cdots + |z_q|^2 + \Re\underbrace{\left[\sum_{j,k=1}^q \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} z_j z_k\right]}_{L} + O\left(|z'| \cdot |z|\right) + O\left(|z|^2\right) \end{aligned}$$

where $z' = (z_{q+1}, \ldots, z_n)$. The essential difference between this and our goal in Equation (1.1) is the nonzero cross terms in the truncated Levi polynomial L. Our task is to find a complex linear isometry u that will diagonalize its matrix.

Take real and imaginary parts of the matrix for the truncated Levi polynomial:

$$\left(\frac{\partial^2\rho(0)}{\partial z_j \partial z_k}\right)_{j,k=1}^{q} = A + iB$$

and define a real $2q \times 2q$ matrix R by $R = \begin{pmatrix} A & -B \\ -B & -A \end{pmatrix}$ (where A and B are real $n \times n$ matrices). For notational purposes, we let $y_j = x_{q+j}$ so that we may also write $z_j = x_j + iy_j$. Also, let $\mathbf{x} = \mathbf{x}(z) = (x_1, \ldots, x_q)$ and $\mathbf{y} = (y_1, \ldots, y_q)$. Since $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ and

$$4\frac{\partial^2\rho(0)}{\partial z_j \partial z_k} = \left(\frac{\partial^2\rho(0)}{\partial x_j \partial x_k} - \frac{\partial^2\rho(0)}{\partial y_j \partial y_k}\right) - i\left(\frac{\partial^2\rho(0)}{\partial x_j \partial y_k} + \frac{\partial^2\rho(0)}{\partial y_j \partial x_k}\right),$$

we see that

$$\begin{aligned}
\Re\left[\frac{\partial^2\rho(0)}{\partial z_j \partial z_k} z_j z_k\right] &= \left(\frac{\partial^2\rho(0)}{\partial x_j \partial x_k} - \frac{\partial^2\rho(0)}{\partial y_j \partial y_k}\right)(x_j x_k - y_j y_k) \\
&\qquad - i\left(\frac{\partial^2\rho(0)}{\partial x_j \partial y_k} + \frac{\partial^2\rho(0)}{\partial y_j \partial x_k}\right)(i x_j y_k + i y_j x_k) \\
&= \left(\frac{\partial^2\rho(0)}{\partial x_j \partial x_k} - \frac{\partial^2\rho(0)}{\partial y_j \partial y_k}\right)(x_j x_k) + \left(\frac{\partial^2\rho(0)}{\partial x_j \partial y_k} + \frac{\partial^2\rho(0)}{\partial y_j \partial x_k}\right)(x_j y_k + y_j x_k) \\
&\qquad - \left(\frac{\partial^2\rho(0)}{\partial x_j \partial x_k} - \frac{\partial^2\rho(0)}{\partial y_j \partial y_k}\right)(y_j y_k)
\end{aligned}$$

$$\Rightarrow \Re\left[\sum_{j,k=1}^{q} \frac{\partial^2\rho(0)}{\partial z_j \partial z_k} z_j z_k\right] = (A\cdot\mathbf{x}, \mathbf{x}) - (iB\cdot\mathbf{x}, i\mathbf{y}) - (iB\cdot\mathbf{y}, i\mathbf{x}) - (A\cdot\mathbf{y}, \mathbf{y}).$$

Thus

$$\Re\left[\sum_{j,k=1}^{q} \frac{\partial^2\rho(0)}{\partial z_j \partial z_k} z_j z_k\right] = (R\cdot\mathbf{x}, \mathbf{x})$$

and

$$\rho(\mathbf{x}) = (\mathbf{x} + R\cdot\mathbf{x}, \mathbf{x}) + O\left(|z'|\cdot|z|\right).$$

Note that R is a real symmetric matrix, so all of its eigenvalues are real. If $\mathbf{x} = \mathbf{x}(e)$ is an eigenvector of R with eigenvalue λ then

$$\mathbf{x}(ie) = (-x_{q+1}, \ldots, -x_{2q}, x_1, \ldots, x_q)$$

is an eigenvector of $\begin{pmatrix} -A & +B \\ +B & +A \end{pmatrix}$ with eigenvalue λ and hence a eigenvector of R with eigenvalue $-\lambda$.

From Chapter 7, Theorem 2.9 in [**1**], there exists an orthonormal basis of eigenvectors $\mathbf{x}_j$ for R. Choose $e_1, \ldots, e_q$ in $\mathbb{C}^q \times (\underbrace{0, \ldots, 0}_{n-q \text{ times}})$, $|e_j| = 1$ so that $\mathbf{x}_j = \mathbf{x}(e_j)$ is an eigenvector of R with eigenvalue $\lambda_j \geq 0$. Since the $\mathbf{x}_j$ are orthogonal, so are the e_j. Complete $\{e_j\}$ to an orthonormal basis for $\mathbb{C}^n$ by letting $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the remaining unit vectors in the directions of the z_j axes when $j > q$.

Let u be the complex linear map defined by $u(z) = \sum z_j e_j$. Then since the e_j form an orthonormal basis for $\mathbb{C}^n$, u is unitary. In terms of $\mathbf{x}$, u is represented by the matrix

$$U := \begin{pmatrix} x_1(e_1) & \ldots & x_1(e_q) & x_1(ie_1) & \ldots & x_1(ie_q) \\ \vdots & & \vdots & \vdots & & \vdots \\ x_q(e_1) & \ldots & x_q(e_q) & x_q(ie_1) & \ldots & x_q(ie_q) \end{pmatrix}$$

Define Λ to be $q \times q$ the diagonal matrix with entries λ_j. Then note that the matrix U was chosen to diagonalize R, so that

$$U^{-1}RU = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_q & & & \\ & & & -\lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_q \end{pmatrix} = \Lambda \oplus (-\Lambda).$$

We have

(1.2)

$$\begin{aligned} \rho(v \circ u(z)) &= (U \cdot \mathbf{x} + RU \cdot \mathbf{x}, U \cdot \mathbf{x}) = \left(U^{-1}U \cdot \mathbf{x} + U^{-1}RU \cdot \mathbf{x}, \mathbf{x}\right) \\ &= (\mathbf{x} + [\Lambda \oplus (-\Lambda)] \cdot \mathbf{x}, \mathbf{x}) = (\mathbf{x}, \mathbf{x}) + ([\Lambda \oplus (-\Lambda)] \cdot \mathbf{x}, \mathbf{x}) \\ &= \sum_{j=1}^{q} |x_j|^2 + \sum_{j=1}^{q} \lambda_j |x_j|^2 - \sum_{j=q+1}^{2q} |x_j|^2 - \sum_{j=1}^{q} \lambda_j |x_{q+j}|^2 \\ &= \sum_{j=1}^{q} (1 + \lambda_j)|x_j|^2 + \sum_{j=q+1}^{2q} (-1 + \lambda_{j-q})|x_j|^2 \end{aligned}$$

with an error of $O\left(|z'| \cdot |z|, |z|^2\right)$. Complete the proof by setting $t = v \circ u$. $\square$

2. Results for q-complete domains

THEOREM 2.1. *Let Ω be a q-complete manifold, $q \geq 1$ with exhaustion function ρ. Let $p \in \Omega$ and X a complex tangent to M at p. Then there exists $F : \Delta \to \Omega$ such that F is proper, $F(0) = p$, and $F'(0) = \lambda X$ for some $\lambda > 0$.*

That is, given any q-complete manifold Ω and any prescribed point $p \in \Omega$, a proper analytic disc passes through p.

Corollary 2.2. *Let Ω be a smoothly bounded q-complete domain in $\mathbb{C}^n$. Let $p \in \Omega$ and X a complex direction. Then there exists $F : \Delta \to \Omega$ such that F is proper, $F(0) = p$, and $F'(0) = \lambda X$ for some $\lambda > 0$.*

Remark 2.3. Since the most difficult case is that where $q = 1$, it suffices to prove the theorem for $q = 1$.

According to Morse's theorem, since Ω is a complex analytic manifold, there exists an exhaustion function for Ω whose set of critical points contains no accumulation point. In particular, one can assume that if p is a critical point of ρ then there exists an open interval I about $\rho(p) \in \mathbb{R}$ such that there exists no critical point q of ρ with $\rho(q) \in I$.

Definition 2.4. From Theorem 5.3.6 in [**10**] and Definition 1.1, there exists a holomorphic, regular, $1-1$ embedding $E : \Omega \to \mathbb{C}^N$ for some N. Let Ω^* be an open neighborhood of $E(\Omega)$ in $\mathbb{C}^N$ with a holomorphic retraction $\pi : \Omega^* \to E(\Omega)$.

Notation. Throughout this section, we will have the following:

- We take Ω, Ω^*, π, and E as in Definition 2.4.
- When $\nabla\rho|_z \neq 0$ we let $e_1 = e_1(z)$ be the unit vector in the direction of $\nabla\rho|_z$. When $\nabla\rho|_z = 0$ it is a unit eigenvector of $\mathcal{L}$ in one of the two or more directions for which the Levi form $\mathcal{L}_\rho(z, e_1)$ for ρ is positive.
- For each $z \in \Omega$, $e_2 = e_2(z)$ is a unit eigenvector of $\mathcal{L}$ in a direction perpindicular to e_1 for which the Levi form $\mathcal{L}_\rho(z, e_2)$ for ρ is positive.
- The symbol Δ denotes the open unit disk in $\mathbb{C}$.
- The symbol Ω_t denotes the level set $\{z \in \Omega \ : \ \rho(z) < t\}$ for $t \in \mathbb{R}$.
- The maps F and G are always of type $F, G : \overline{\Delta} \to \Omega$, continuous on $\overline{\Delta}$, and analytic on Δ.
- If f is a map, and a set S is in the domain of f, then $f(S)$ denotes the set $\{z \in \text{Range}(f) \ : \ f(s) = z \text{ for some } s \in S\}$. If the range of f is a subset of $\mathbb{R}$, then we write $a < f(S) < b$ if and only if $a < f(s) < b$ for all $s \in S$.

Definition 2.5. Two maps F and G are said to *match* if $F(0) = G(0)$ and $F'(0) = G'(0)$.

Definition 2.6. Given a q-complete manifold Ω with exhaustion function ρ and embedding E, we say that G is (ϵ, R)-*close* to F if G matches F, $\rho(G(z)) - \rho(F(z)) > -\epsilon$ for all $z \in \Delta$, and $|E(G(z)) - E(F(z))| < \epsilon$ for all $z \in \Delta$ with $|z| < R$.

Lemma 2.7. *If $0 \notin Crit(\rho)$, then there exists another function $\tilde{\rho}$ such that*

$$\partial\Omega_0 = \{z \ : \ \tilde{\rho}(z) = 0\}$$

and such that for all z in a neighborhood of $\partial\Omega_0$, both $\mathcal{L}_{\tilde{\rho}}(z, e_2)$ and $\mathcal{L}_{\tilde{\rho}}(z, e_1)$ are strictly positive. That is, $\tilde{\rho}$ is 2-subharmonic near $\partial\Omega_0$.

Proof. Consider functions of the form

$$\tilde{\rho}(z) = \rho(z) + A\rho^2(z)$$

for constant $A > 0$. Take $z \in \partial\Omega_0$. We have

$$\begin{aligned} \mathcal{L}_{\tilde{\rho}}(z, e_2) &= \mathcal{L}_{\rho}(z, e_2) + 2A\rho\mathcal{L}_{\rho}(z, e_2) + 2A|\nabla\rho \cdot e_2|^2 \\ &= \mathcal{L}_{\rho}(z, e_2) \end{aligned}$$

since $\rho = 0$ on $\partial\Omega_0$ and e_2 is chosen such that $\nabla\rho \cdot e_2 = 0$. Also,

$$\begin{aligned} \mathcal{L}_{\tilde{\rho}}(z, e_1) &= \mathcal{L}_{\rho}(z, e_1) + 2A\rho\mathcal{L}_{\rho}(z, e_1) + 2A|\nabla\rho \cdot e_1|^2 \\ &= \mathcal{L}_{\rho}(z, e_1) + 2A|\nabla\rho|^2 \end{aligned}$$

so by choosing A such that

$$A > \max_{z \in \partial\Omega_0} \left| \frac{\mathcal{L}_{\rho}(z, e_1)}{2|\nabla\rho|^2} \right|$$

we obtain a new defining function of Ω_0 in a neighborhood of $\partial\Omega_0$ with positive Levi form in the directions e_1 and e_2. $\square$

The following extension of a fact from [**7**] allows us to "trim back" an analytic disk to a level set of ρ. That is, the intersection D of an analytic disk with a level set of ρ is itself an analytic disk, by virtue of the fact that we can use the Riemann mapping theorem to map Δ into the preimage of D. Thus, this lemma will allow us to assume that the boundaries of our analytic disks lie on level sets for ρ. In light of Lemma 2.7, and the fact that the rest of our lemmas are local with respect to level sets of ρ, we can assume in the rest of this section that $\mathcal{L}_{\rho}(z, e_1) > 0$.

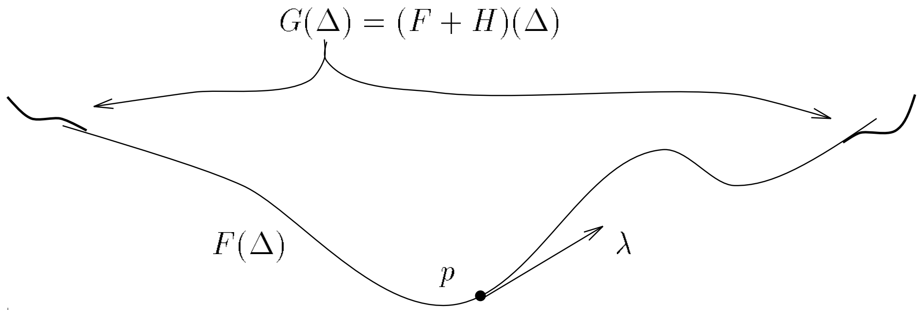

FIGURE 2.1. "Pulling" the boundary of an analytic disk

LEMMA 2.8. *Given an analytic map $F : \overline{\Delta} \to \Omega$ with $\rho(F(0)) = a$, and a real number $b > a$, there exists an analytic map $G : \overline{\Delta} \to \Omega$ equal to F on $F^{-1}(\Omega_b)$ such that $G(\partial\Delta) \subset \overline{\Omega}_b$. If $\rho(F(\partial\Delta)) > b$, then $\rho(G(\partial\Delta)) = b$.*

PROOF. The first paragraph of the proof of Lemma 1 in [**7**] may be used verbatim. □

We now want to construct the basic process for "pulling" the boundary of $F(\overline{\Delta})$ toward the boundary of Ω (Figure 2.1). The idea is that, as long as we are increasing ρ on the boundary of the image $F(\partial\Delta)$, it is getting closer to the boundary of Ω. How might we increase ρ? Observe that in local coordinates the Taylor series for ρ about a point p_0 is

$$\rho(p) = \rho(p_0) + L_{p_0}(p - p_0) + \mathcal{L}(p_0, p - p_0) + o(|p - p_0|^2).$$

If we consider ρ restricted to the Levi support surface $\{L_{p_0}(z) = 0\}$, then

$$\rho(p) = \rho(p_0) + \mathcal{L}(p_0, p - p_0) + o(|p - p_0|^2).$$

Assume that we know $p - p_0$ lies in one of the directions for which the Levi form $\mathcal{L}(p_0, p - p_0)$ is strictly positive. Then we obtain, for some $c > 0$, $\mathcal{L}(p_0, p - p_0) > c|p - p_0|^2$, or

$$\rho(p) > \rho(p_0) + \frac{1}{2}c|p - p_0|^2 \tag{2.1}$$

for p sufficiently close to p_0.

Although we cannot directly choose a point p to which to pull $\partial\Delta$, we can nevertheless use this idea to accomplish our actual goal: to increase ρ on $\partial\Delta$. A small disk image about p lying simultaneously in the Levi support surface and in a direction for which $\mathcal{L}$ is positive would have ρ larger on its boundary than at the origin. Simply by knowing that we had pulled $\partial\Delta$ to *any* boundary point of that disk, we would know that we had increased ρ. Equation (2.1) shows that the amount of this increase depends (when p is close to p_0) only on the radius of the disk.

By placing small analytic disk images Δ_ζ all around the boundary $F(\partial\Delta)$, and pulling to their respective boundaries, we obtain a method for increasing ρ at the boundary of Δ. Below, Lemma 2.9 provides the "pulling" technique, while Lemma 2.10 provides a way of placing the disk images at boundary points. What obstructions can we find to the size of the increase of ρ? (This is equivalent to asking what limits the radii of the disk images Δ_ζ.)

First of all, we need the second-degree approximation ignoring the error $o(|p-p_0|^2)$ to be accurate all around the boundary of the disk images. The error is a function that depends on the third and higher derivatives of ρ, and can be bounded, say, in any compact subset of Ω. Second, technical limitations in Lemma 2.9 limit the size of the disk images depending on the embedding E of Ω and the sizes of coordinate neighborhoods for Ω.

Finally, and most vital to our analysis, is the requirement that the disk images lie in Levi support surfaces. For the disk images to be chosen smoothly means that they must be 1-dimensional manifolds, so the Levi support surface in which they lie should themselves be regular manifolds. So long as p_0 is not a critical point of ρ, this is possible – locally, at least. However, the Levi support surfaces cease to be regular manifolds as they approach a critical point, so the maximum size of a disk image at p_0 is limited by the proximity of p_0 to a critical point of ρ.

This last obstruction illustrates the basic dichotomy between regular points and critical points that affects our construction. Disks can be pulled using the method outlined above, so long as no critical points are nearby. The problem of increasing ρ near a critical point, though, will the require special treatment in Lemmas 2.13 through 2.18.

Lemma 2.9. *Suppose F is as above, with constants $\delta, \epsilon > 0$ and $0 < R < 1$, and a collection of smooth holomorphic maps $\Delta_\zeta : \overline{\Delta} \to \Omega$ such that Δ_ζ varies smoothly with $\zeta \in \partial\Delta$ with $\Delta_\zeta(0) = F(\zeta)$. Suppose also that we have r_0 such that each disk* *image* $\Delta_\zeta(\overline{\Delta})$ *satisfies the* **size condition**

$$|E(F(\zeta)) - E(\Delta_\zeta(w))| < r_0.$$

for all $\zeta \in \partial\Delta$ *and* $w \in \overline{\Delta}$*, with* r *so small that for all* $z \in E(F(\overline{\Delta}))$, $\mathbf{B}_{r_0}(z) \subset\subset \Omega^*$.

Then there exists a holomorphic $G : \overline{\Delta} \to \Omega$ *which is* (ϵ, R)*-close to* F*, so that for each* $\zeta \in \partial\Delta$*, there exists* $\xi \in \partial\Delta$ *such that*

$$|E(G(\zeta)) - E(\Delta_\zeta(\xi))| < \delta.$$

Proof. Define

$$H(\zeta, w) : \partial\Delta \times \overline{\Delta} \longrightarrow \mathbb{C}^N$$

by

$$H(\zeta, w) = E\left(\Delta_\zeta(w)\right) - E\left(F(\zeta)\right).$$

Let

$$\sum_{j=0}^{\infty} \mathbf{a}_j(\zeta) w^j$$

be the Weierstrass polynomial for H. Note that $H(\zeta, 0) = 0$, so $\mathbf{a}_0 = 0$. For $m \in \mathbb{N}$, we define

$$H_m(\zeta, w) = \sum_{j=1}^{m} \mathbf{a}_j(\zeta) w^j$$

and we let

$$\eta(m) = \max_{\partial\Delta \times \overline{\Delta}} |H(\zeta, w) - H_m(\zeta, w)|.$$

Observe that $\lim_{m\to\infty} \eta(m) = 0$, so that this cutoff approximation can be made arbitrarily accurate. Note that on $\partial\Delta$, $\overline{\zeta} = 1/\zeta$, so for each j we choose polynomials $P_j^1 \ldots P_j^N$ and $Q_j^1 \ldots Q_j^N$ such that for

$$\mathbf{P}_j\left(\zeta\right) = \left(P_j^1\left(\zeta\right), \ldots, P_j^N\left(\zeta\right)\right)$$

and

$$\mathbf{Q}_j\left(1/\zeta\right) = \left(Q_j^1\left(1/\zeta\right), \ldots, Q_j^N\left(1/\zeta\right)\right)$$

we have

$$|\mathbf{a}_j(\zeta) - \mathbf{P}_j(\zeta) - \mathbf{Q}_j(1/\zeta)| < \frac{\eta(m)}{m}.$$

Let $\ell \in \mathbb{N}$ be greater than the degree of any of the Q_j^k, and define the polynomial

$$f_m(\zeta) = \sum_{j=1}^{m} \left[\mathbf{P}_j(\zeta) + \mathbf{Q}_j(1/\zeta)\right] \zeta^{j\ell}.$$

Then since ℓ is so large, no negative powers of ζ occur in f_m, hence f_m is defined (and bounded) for all $\zeta \in \Delta$.

We also have

$$\left|f_m(\zeta) - H(\zeta, \zeta^\ell)\right| < \eta(m) + m\frac{\eta(m)}{m} = 2\eta(m), \tag{2.2}$$

for $\zeta \in \partial\Delta$, so the f_m form an arbitrarily close approximation to H. f_m will give the "bump" function we add to F, so we also want to make sure that it changes F very little inside $|z| < R$ in order to keep our new "bumped" analytic disk (ϵ, R)-close to $F(\Delta)$. Take γ to be so small that for $z \in E(F(\Delta))$, we have

$$\left|\rho(F(z)) - \rho \circ E^{-1} \circ \pi\left(E(F(z)) + \gamma t\right)\right| < \epsilon \qquad \text{for all } t \in \mathbf{B}^N \subset \mathbb{C}^N$$

By choosing ℓ sufficiently large, we ensure that

$$f_m(z) < \gamma \qquad \text{for all } |z| < R. \tag{2.3}$$

and that $f(0) = f'(0) = 0$. Equation (2.2) implies

$$\left|\pi\left(E(F(\zeta)) + f_m(\zeta)\right) - E(F(\zeta)) - H(\zeta, \zeta^\ell)\right| \to 0 \quad \text{ as } m \to \infty$$

for $\zeta \in \partial\Delta$. Choose m so large that

$$\left|\pi\left(E(F(\zeta)) + f_m(\zeta)\right) - E(F(\zeta)) - H(\zeta, \zeta^\ell)\right| < \delta$$

for all $\zeta \in \partial\Delta$. We define

$$G(z) = E^{-1} \circ \pi\left(E(F(z)) + f_m(z)\right).$$

Then Equation (2.3) implies that G is (ϵ, R)-close to F, and we have

$$\begin{aligned} |E(G(\zeta)) - E(\Delta_\zeta(\xi))| &= |\pi\left(E(F(\zeta)) + f_n(\zeta)\right) - \left(E(F(\zeta)) + H(\zeta, \zeta_n)\right)| \\ &< \delta \end{aligned}$$

for $\xi = \zeta^\ell$, as required. □

Now we provide a method for creating disk images for use as input to Lemma 2.9.

Lemma 2.10. *Given F with $\rho(F(\partial\Delta)) \cap Crit(\rho) = \emptyset$, there exist a constant $\alpha > 0$ and holomorphic maps $\Delta_\zeta : \overline{\Delta} \to \Omega$ such that Δ_ζ varies smoothly with $\zeta \in \partial\Delta$, $\Delta_\zeta(0) = F(\zeta)$, and for all $\zeta, \xi \in \partial\Delta$,*

$$\rho\left(\Delta_\zeta(\xi)\right) > \rho\left(\Delta_\zeta(0)\right) + \alpha.$$

Furthermore, we can assume that the Δ_ζ satisfy the size condition in Lemma 2.9.

PROOF. For $\zeta \in \partial\Delta$, define a manifold in which to place an analytic disk image by

$$S_\zeta := \langle e_1, e_2 \rangle \cap \{z \in \Omega \ : \ L_{F(\zeta)}(z) = 0\}$$

where L denotes the Levi polynomial. Note that S_ζ is generically 1-dimensional, so by taking a small perturbation of L if necessary, we can assume S_ζ is indeed 1-dimensional. Also, since there are no critical points near $F(\partial\Delta)$, we can assume that the S_ζ vary smoothly in ζ – at least near $F(\partial\Delta)$.

Let

$$c = \min_{\substack{i=1,2\\ \zeta\in\partial\Delta}} [\mathcal{L}_\rho(F(\zeta), e_i)]$$

and note that $c > 0$ by our choice of e_1 and e_2.

Choose a neighborhood M of ζ so that the Levi polynomial L has no critical points in M, and a defining function $s : \mathbb{C} \to \Omega$ for S_ζ in M such that $s(0) = F(\zeta)$. Note that for each ζ, L is holomorphic, so s is holomorphic as well. By our previous discussion, we know that since S_ζ is in the Levi support surface for ρ, we can shrink M (if necessary) to the point that for any $a, b \in \mathbb{R}$ where $F(\zeta) + ae_1 + be_2 \in M$,

$$\rho(F(\zeta) + ae_1 + be_2) > \rho(F(\zeta)) + \frac{c}{2}\left(|a|^2 + |b|^2\right).$$

Let $D \subset \mathbb{C}$ be the largest disc (say of radius r) centered at the origin whose image under the map s is contained in M and satisfies the size condition in Lemma 2.9. Define $\Delta_\zeta(z) = s(rz)$.

Given $\xi \in \partial\Delta$, we find $a_\xi^\zeta, b_\xi^\zeta \in \mathbb{C}$ so that

$$\Delta_\zeta(\xi) = s(r\xi) = \zeta + a_\xi^\zeta e_1 + b_\xi^\zeta e_2.$$

Now let ζ vary, and define

$$m = \min_{\zeta,\xi\in\partial\Delta}\left[\left|a_\xi^\zeta\right|^2 + \left|b_\xi^\zeta\right|^2\right].$$

By compactness, $m > 0$ and the choice of M ensures that

$$\rho\left(\Delta_\zeta(\xi)\right) > \frac{mc}{2} + \rho\left(\Delta_\zeta(0)\right).$$

Thus we define $\alpha = \frac{mc}{2}$. $\square$

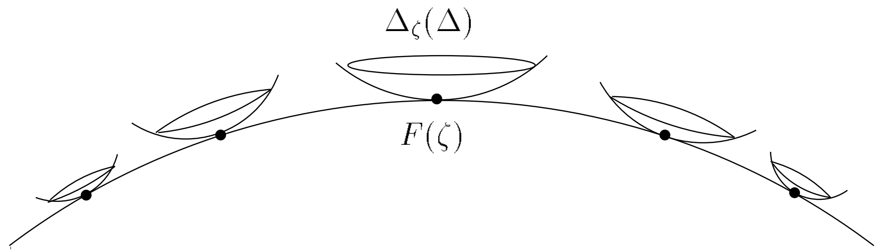

FIGURE 2.2. The analytic disks Δ_ζ along the boundary $F(\Delta)$

Combining Lemmas 2.9 and 2.10 gives the basic method for pulling the disk boundary when no critical points are involved.

LEMMA 2.11. *Given F as above, $\epsilon > 0$ and $0 < R < 1$ such that*

$$\rho(F(\partial\Delta)) \cap Crit(\rho) = \emptyset,$$

there exists an $\alpha > 0$ independent of ϵ and R and an analytic map G which is (ϵ, R)-close to F with

$$\rho(G(\zeta)) > \rho(F(\zeta)) + \alpha/2$$

for all $\zeta \in \partial\Delta$.

PROOF. Use Lemma 2.10 to create α and Δ_ζ. Choose δ sufficiently small that whenever

$$|z - w| < \delta \quad \text{for } z, w \in E(\Omega)$$

we have

$$\left|\rho\left(E^{-1}(z)\right) - \rho\left(E^{-1}(w)\right)\right| < \alpha/2.$$

Lemma 2.9 then provides G. $\square$

REMARK 2.12. The size α of the increase in $\rho(\partial\Delta)$ obtained by Lemma 2.11 depends on two things – both arising in the proof of Lemma 2.10: first, the size of the Levi form in the e_1 and e_2 directions affects how quickly ρ increases on the disks $\Delta_\zeta(\Delta)$; second, the distance of $F(\partial\Delta)$ from critical points of ρ determines the radii of those disks. Our increase α goes as the product of these quantities.

We are now ready to address the problem of pulling the boundary of our analytic disk in a region near a critical point. We begin with a lemma that shows we can get as close to the critical value as necessary.

LEMMA 2.13. *Given F, a critical value c of ρ, $\epsilon > 0$, $0 < R < 1$, constants $a < b < c$ such that $\nabla\rho \neq 0$ in $\Omega_c \setminus \Omega_a$, and F such that $F(\partial\Delta) \subset \Omega_c \setminus \Omega_b$ then for all $d < c$ there exists a G which is (ϵ, R)-close to F such that $\rho(G(\partial\Delta)) = d$.*

PROOF. Take c' such that $d < c' < c$, and in Lemma 2.10 choose the radii of the analytic discs Δ_ζ so small that $\rho(\Delta_\zeta(\xi)) \leq c'$ for all $\zeta, \xi \in \partial\Delta$. Use these as input to Lemma 2.9, along with δ so small that $\rho(G(z)) < \frac{c+c'}{2}$. Repeat this procedure if necessary until $\rho(F(\partial\Delta)) > d$. Finally, use Lemma 2.8 to trim back G so that $\rho(G(\partial\Delta)) = d$. $\square$

Now we need to show that the analytic disk boundary can be pulled *past* the critical value in regions away from a critical point. The idea here is to choose the disk images Δ_ζ as normal outside a small neighborhood of the critical point, while choosing them to be infinitesemally small near the critical point. Away from the critical point, we can increase ρ an essentially fixed amount (see Remark 2.12), so we can pull past the critical value. By doing so, we reduce the problem of pulling the disk boundary beyond a critical value to a *local* problem of pulling it past the critical value in a neighborhood of the critical point.

LEMMA 2.14. *Suppose F is as above, and we have a critical point z_0 of ρ with critical value c, $\epsilon > 0$, $0 < R < 1$, and constants $a < b < c$ such that $\nabla\rho \neq 0$ in $\Omega_c \setminus \Omega_a$. Assume also that $F(\partial\Delta) \subset \Omega_c \setminus \Omega_b$. Then there exist a G which is (ϵ, R)-close to F and an open arc $A \subset\subset \partial\Delta$ such that $G(A^c) \subset\subset N_{z_0}$ for a coordinate neighborhood N_{z_0}, and such that $\rho(G(A)) > c$.*

PROOF. Choose an open neighborhood $U \subset\subset N_{z_0}$ of z_0 small enough that $F(\partial\Delta) \not\subset U$, and let $\eta = \operatorname{dist}(U, \partial N_{z_0})$. Let

$$\tilde{\Omega} = \overline{\Omega}_c \setminus \Omega_b \setminus N_{z_0}.$$

As in Lemma 2.10, we define

$$S_p := \langle e_1, e_2 \rangle \cap \{z \in \Omega \ : \ L_p(z) = 0\},$$

for all $p \in \overline{\Omega}_c \setminus \Omega_b$, and choose analytic disks $D_p(\Delta) \subset S_p$. This time, choose the disks with the additional criterion that if $p \notin U$, then $D_p(\Delta) \cap U = \emptyset$. For $p \in U$, just let $D_p(\Delta) \equiv p$. We will see that increasing ρ using these disks will yield the desired analytic disk G.

The set $\tilde{\Omega}$ is compact, and bounded away from any critical points, so for all $p \in \tilde{\Omega}$ there is a global minimum α for the quantity $\rho\left(D_p(\partial\Delta)\right) - \rho\left(p\right)$. This α depends on three things: η; the size of $\mathcal{L}_\rho(\tilde{\Omega}, e_i)$; and the distance between Ω_c and the next critical point whose critical value is larger than c.

By Lemma 2.14, we can assume the map F has the property that

$$\rho\left(F\left(\partial\Delta\right)\right) > c - \alpha/4.$$

For $\zeta \in \partial\Delta$, define $\Delta_\zeta = D_{F(\zeta)}$. Apply Lemma 2.9 to the disks Δ_ζ with the constant δ chosen so small that whenever

$$|z - w| < \delta \quad \text{ for } z, w \in E(\Omega)$$

we have

$$\left|\rho\left(E^{-1}(z)\right) - \rho\left(E^{-1}(w)\right)\right| < \alpha/2.$$

Let G be the resulting map, and define

$$A = F^{-1}\left(N_{z_0}^c\right) \cap \partial\Delta.$$

Then for $\zeta \in A$,

$$\rho\left(G\left(\zeta\right)\right) > c + \alpha/4$$

as required. $\square$

Here we have a technical lemma generalizing the above.

LEMMA 2.15. *Take F with $F(\partial\Delta) \cap Crit(\rho) = \emptyset$, constants $\delta, \epsilon > 0$, $0 < R < 1$, and arcs*

$$U \subset\subset V \subset\subset \partial\Delta.$$

Then there exists $\alpha > 0$ and G which is (ϵ, R)-close to F such that

$$|E\left(G(\zeta)\right) - E\left(F(\zeta)\right)| < \delta$$

for $\zeta \in U$, and

$$\rho(G(\zeta)) > \rho(F(\zeta)) + \alpha$$

for $\zeta \in V^c$. α is independent of δ, ϵ, and $\rho(z)$ for z in a small neighborhood of $F(U)$.

PROOF. The proof here is exactly like that of Lemma 2.14. Simply substitute V for N_{z_0}. $\square$

We are finally ready to create a method for pulling past a critical value in the neighborhood of a critical point. According to the following analysis, Lemma 1.2 provides a standard, coordinatised form of ρ to work with. Clearly we can assume that we have holomorphic coordinates in a neighborhood of z_0 such that $z_0 = 0$. Lemma 2.7 allows us to assume ρ is 2-convex. Thus we see from Equation (1.2) in the proof of Lemma 1.2 that there exists a coordinate transformation such that

$$\begin{aligned}\rho(z) = (1+\lambda_1)|x_1|^2 + (1+\lambda_2)|x_2|^2 \\ - (1-\lambda_3)|x_3|^2 - (1-\lambda_4)|x_4|^2 + O\left(|z'|\cdot|z| + |z_1|^3 + |z_2|^3\right).\end{aligned}$$

Scaling each coordinate by a factor of $1/(1+\lambda_j)$ and relabelling x_3 and x_4 as y_1 and y_2 yields the standard form we require:

$$\rho(z) = x_1^2 + x_2^2 - c_1 y_1^2 - c_2 y_2^2 + o(|z_1|^2, |z_2|^2) + O(|z_3|, \ldots, |z_N|)\cdot O(|z|)$$

for $c_j = (1-\lambda_j)/(1+\lambda_j)$.

In the proof below, we use the fact that $F(\partial\Delta)$ has not yet quite reached the critical point to pull it in a succession of directions leading *away* from the critical point. These directions are provided by a series of functions ρ_j which are similar to, but not equal to, the original ρ. Naturally, analyticity of the map G doesn't depend on the function according to which we have "pulled" F, but it *is* necessary to choose the ρ_j in such a way that G will indeed pull the boundary of our disk past the critical value c.

LEMMA 2.16. *Suppose z_0 is a critical point of ρ with critical value 0. Assume also that there is an arc $V \subset\subset N_{z_0}$ such that*

$$\rho\left(F(\partial\Delta)\setminus V^c\right) > \beta > 0$$

where N_{z_0} denotes a coordinate neighborhood of z_0 with coordinate function ψ Then given $\epsilon > 0$ and $0 < R < 1$, there exists G which is (ϵ, R)-close to F such that $\rho\left(G(\partial\Delta)\right) > 0$.

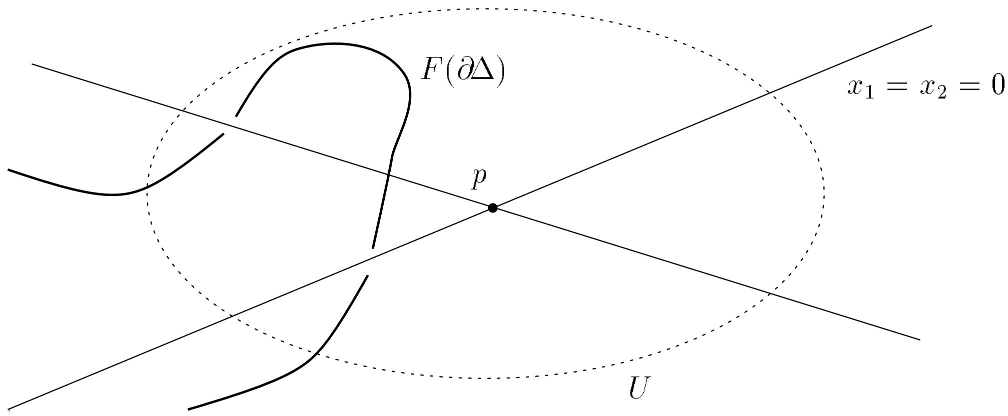

FIGURE 2.3. The map $F(\partial\Delta)$ misses the axis

PROOF. Let $W \subset N_{z_0}$ such that $V \subset\subset W$, and find open sets V_0, U_i, and V_i with

$$W \supset\supset V_0 \supset\supset V_1 \supset\supset U_1 \supset\supset V_2 \supset\supset U_2 \supset\supset \dots$$

and

$$V \subset\subset \bigcap_{i \in \mathbb{N}} V_i.$$

Choose (ϵ_i, R_i) so that $\sum_{i=1}^{\infty} \epsilon_i < \epsilon$ and $R < R_i \nearrow 1$ and assume without loss of generality that $\epsilon \ll \beta$. Take holomorphic coordinates z_i in N_{z_0} such that

$$\mathcal{L}\left(z_0, \frac{\partial}{\partial z_1}\right) > \mathcal{L}\left(z_0, \frac{\partial}{\partial z_2}\right) > 0.$$

Then according to Lemma 1.2, we can choose coordinates in N_{z_0} so that $z_0 = 0$ and

$$\rho(z) = x_1^2 + x_2^2 - c_1 y_1^2 - c_2 y_2^2 + f(z) + g(z)$$

for some bounded, continuous $f(z) = o(|z_1|^2, |z_2|^2)$ and $g(z) = O(|z_3|, \dots, |z_N|) \cdot O(|z|)$.

Note that $\dim_{\mathbb{R}}(\partial\Delta) = 1$ so that $F(\partial\Delta)$ generically misses the axis $x_1 = x_2 = 0$. Thus by taking (if necessary) a small perturbation of the map F we can assume that there exists a constant $\eta > 0$ such that $x_1^2 + x_2^2 > 2\eta$ for all points in $F(\partial\Delta)$. Assume without loss of generality that $2\eta \ll \beta$, and choose constants $\delta_i > 0$ such that $\sum \delta_i < \delta$ for a δ so small that whenever

$$|z - w| < \delta \quad \text{for } z, w \in E(\Omega)$$

we have

$$\left|\rho(E^{-1}(z)) - \rho(E^{-1}(w))\right| < \frac{\eta}{4} \min_{z \in W}(1, |Dg(z)|^{-1}). \tag{2.4}$$

We now use a trick to get past the critical value. The preceding lemmas do not depend on ρ being an exhaustion function, but rather only on the fact that there are two positive directions for the Levi form of ρ. Consider such a function

$$\rho_1(z) = x_1^2 + x_2^2 - \eta y_1^2 - \eta y_2^2.$$

Note that $w \in F(\partial\Delta) \Rightarrow \rho_1(w) > 0$ so that we can combine Lemmas 2.15 and 2.11 with our choice of $e_1 = \frac{\partial}{\partial z_1}$, $e_2 = \frac{\partial}{\partial z_2}$ to obtain a new F_1 which is (ϵ_1, R_1)-close to F such that for each $\zeta \in \partial\Delta$, there exists $\xi \in \partial\Delta$ with

$$|E(F_1(\zeta)) - E(\Delta_\zeta(\xi))| < \delta_1,$$

as well as the properties that

$$\rho\left(F_1(\zeta)\right) - \rho\left(F(\zeta)\right) > -\epsilon_1$$

and

$$\rho_1\left(F_1(\zeta)\right) - \rho_1\left(F(\zeta)\right) > -\epsilon_1$$

for $\zeta \in F^{-1}(V_1^c) \cap \partial\Delta$. Most importantly,

$$\rho_1\left(F_1(\zeta)\right) > 2\eta$$

for $\zeta \in F_1^{-1}(U_1) \cap \partial\Delta$. We have now improved our situation to the point that

$$\rho_2(\zeta) := x_1^2 + x_2^2 - 2\eta y_1^2 - 2\eta y_2^2 > 0$$

when $\zeta \in F^{-1}(U_1) \cap \partial\Delta$.

Repeat the process (using Lemmas 2.15 and 2.11) we used to obtain F_1 from ρ_1 above, this time using the function ρ_2 and the map F_1 as initial data. We obtain F_2 which is (ϵ_2, R_2)-close to F_1 such that for each $\zeta \in \partial\Delta$, there exists $\xi \in \partial\Delta$ with

$$|E(F_2(\zeta)) - E(\Delta_\zeta(\xi))| < \delta_2,$$

as well as the properties that

$$\rho\left(F_2(\zeta)\right) - \rho\left(F_1(\zeta)\right) > -\epsilon_2$$

and

$$\rho_2\left(F_2(\zeta)\right) - \rho_2\left(F_1(\zeta)\right) > -\epsilon_2$$

for $\zeta \in F_2^{-1}(V_2^c) \cap \partial\Delta$. Also,

$$\rho_2\left(F_2(\zeta)\right) > 2\eta$$

for $\zeta \in F^{-1}(U_2) \cap \partial\Delta$. We can repeat the process n times, so long as $n\eta < 1$ (that is, so long as ρ_n is strictly 2-subharmonic).

Choose n sufficiently large that $1 > n\eta > \max[c_1, c_2]$ to get

$$x_1^2 + x_2^2 - y_1^2 - y_2^2 < \rho_n(z) = x_1^2 + x_2^2 - n\eta y_1^2 - n\eta y_2^2 < \rho(z) - g(z)$$

near z_0. We obtain F_n which is $(\epsilon/2, R)$-close to F and has $\rho_n(F_n(\partial\Delta) \cap U_n) > 2\eta$. Note that since we always chose our disks Δ_ζ in the e_1, e_2 directions, the function $g(z)$ can be seen from equation (2.4) and our choice of the δ_i to satisfy

$$|g(F_n(\zeta)) - g(F(\zeta))| < \eta.$$

Thus F_n has

$$\rho(F_n(\zeta)) - \rho(F(\zeta)) > \beta - \epsilon > 0$$

for $\zeta \in F^{-1}(W^c) \cap \partial\Delta$, and

$$\begin{aligned} \rho(F_n(\zeta)) &> \rho(F_n(\zeta)) - g(\zeta) \\ &> \rho_n(F_n(\zeta)) - \eta \\ &> \eta > 0 \end{aligned}$$

for $\zeta \in F_n^{-1}(U_n) \cap \partial\Delta$.

We now have that $\rho(F_n(\zeta)) > 0$ for all $\zeta \in \partial\Delta \backslash F^{-1}(W \setminus V)$. The set $F^{-1}(W \setminus V)$ is contained in two arcs which are removed from a neighborhood of the critical point, so we simply use Lemma 2.15 to obtain G from F_n such that G is(ϵ, R)-close to F, with $G(\partial\Delta) > 0$. $\square$

We are now ready to prove that it is possible to pull the boundary of our analytic disk completely past a critical point.

Lemma 2.17. *Suppose z_0 is a critical point of ρ with critical value* 0*, and for arbitrarily small constants $a < b < 0$ there is F with $F(\partial\Delta) \subset \Omega_b \setminus \overline{\Omega}_a$. Then for one such F, and any $\epsilon > 0$ and $0 < R < 1$ there exists G (ϵ, R)-close to F such that $\rho(G(\partial\Delta)) > 0$.*

Proof. Use Lemma 2.14 to create the input for Lemma 2.16. The resulting G is the desired function. $\square$

The final lemma sets up the framework for the limiting sequence of analytic disks used in the proof of the theorem.

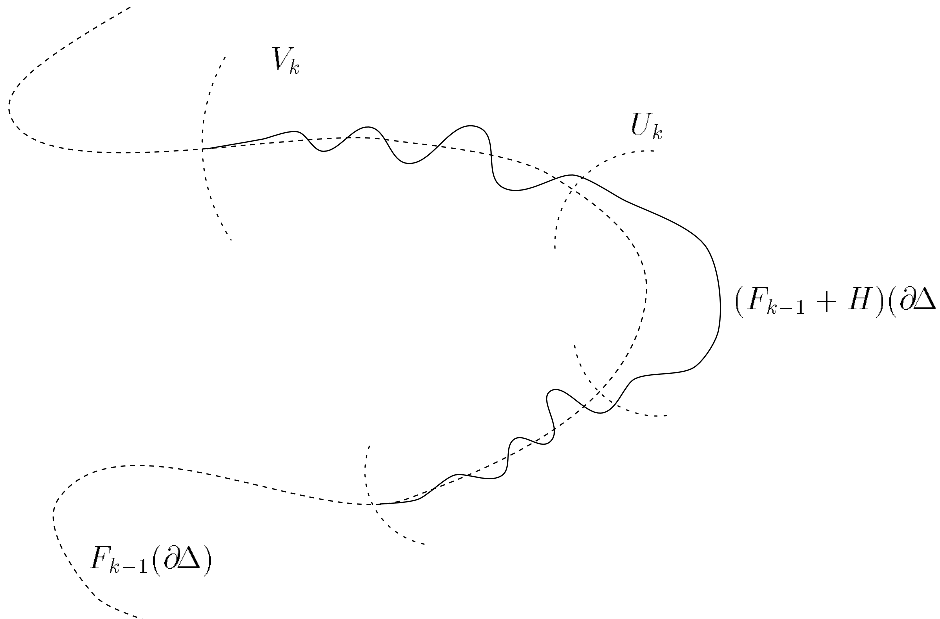

FIGURE 2.4. The succession of maps increasing $\rho\,(F\,(\partial\Delta))$

LEMMA 2.18. *Given $N \in \mathbb{N}$, $\epsilon > 0$ and $0 < R < 1$, and F such that $F(\Delta) \subset \Omega_N$, there exists G which is (ϵ, R)-close to F such that $\rho\,(G(\partial\Delta)) > N$.*

PROOF. Let (ϵ_i, R_i) be chosen so that $\sum_{i=1}^{\infty} \epsilon_i < \epsilon$ and $R < R_i \to 1$. Assume without loss of generality that $\rho\,(G(\partial\Delta)) = 0$, and let the critical values for ρ between 0 and N be $\{c_i\}_{i=1}^{n}$. Use Lemma 2.11 a finite number, say k_1 times to obtain a map F_{k_1} and a critical point c_1 suitable for input to Lemma 2.17. Apply Lemma 2.17 to obtain F_{k_1+1}, and proceed to apply Lemma 2.11 k_2 times to obtain F_{k_2+1}, c_2 suitable for input to Lemma 2.17. Continuing the process, we end up with the required map $G = F_{n+\sum k_i}$. □

Proof of Theorem 2.1 We want to show that it is possible to use the lemmas above to create a convergent sequence of holomorphic maps from Δ into Ω whose limit is proper. Begin by finding a $\delta > 0$ so small that whenever

$$|z - w| < \delta \quad \text{for } z \in E(\Omega) \text{ and } w \in \mathbb{C}^n$$

we have

$$w \in \Omega^*.$$

Without loss of generality, we can assume that $\delta < 1$.

Choose (ϵ_i, R_i) so that $\sum \epsilon_i < \delta$, and $R_i \nearrow 1$. Take a coordinate neighborhood (M, ψ) of our prescribed point p with $\psi(p) = 0$, and put a small linear disk $F : \overline{\Delta} \to \psi(M)$ in $\psi(M)$ in such a way that $F'(0) = \psi_*(\lambda)$. For the succession of $i \in \mathbb{N}$, use Lemma 2.18 and Lemma 2.8 to construct F_i such that $\rho(F_i(\partial\Delta)) = i$, and F_i is (ϵ_i, R_i)-close to F_{i-1}.

We want to prove that the sequence F_i converges uniformly on compact subsets of Δ. First, we need to see that the pointwise limit exists for any $z \in \Delta$. Let $k \in \mathbb{N}$ be large enough that $|z| < R_k$, and note that by the Definition 2.6 of (ϵ, R)-closeness,

$$|E\left(F_{i+1}\left(z\right)\right) - E\left(F_i\left(z\right)\right)| < \epsilon_i$$

for $i \geq k$. Hence

$$|E\left(F_{i+\ell}\left(z\right)\right) - E\left(F_i\left(z\right)\right)| < \sum_{j=i}^{i+\ell} \epsilon_j < \delta.$$

Therefore the sequence $E\left(F_i\left(z\right)\right)$ is bounded, and hence has a limit z_∞. By regularity of E, and the fact that $E\left(F_i\left(z\right)\right) \in E(\Omega)$ for all i, we see that $z_\infty \in E(\Omega)$. Define the pointwise limit function F of the F_i by $F(z) := E^{-1}\left(z_\infty\right)$.

Let $K \subset\subset \Delta$ and $\epsilon > 0$. Define

$$\delta_j = \sum_{i=j}^{\infty} \epsilon_i$$

and note that $\delta_j \to 0$ as $j \to \infty$. Since K is relatively compact in Δ, there exists a $k_1 \in \mathbb{N}$ such that

$$K \subset \left\{z \ : \ |z| < R_{k_1}\right\}.$$

Choose $k_2 \in \mathbb{N}$ such that $\delta_{k_1} < \epsilon$, and define $k = \min\left[k_1, k_2\right]$. Then for all $z \in K$ and $\ell > k$,

$$\begin{aligned} |E\left(F_\ell\left(z\right)\right) - E\left(F\left(z\right)\right)| &\leq \sum_{j=\ell}^{\infty} |E\left(F_j\left(z\right)\right) - E\left(F_{j+1}\left(z\right)\right)| \\ &< \sum_{j=\ell}^{\infty} \epsilon_j \\ &< \delta_\ell < \delta_k \\ &< \epsilon. \end{aligned}$$

Hence $F_i \to F$ uniformly on compact subsets of Δ. Since the F_i are holomorphic, so is F. Our only remaining requirement is to show that F is proper.

Take a sequence $\{z_j\}$ in Δ with $|z_j| \nearrow 1$. Given $M \in \mathbb{N}$, we want to show that there exists $k \in \mathbb{N}$ such that

$$\rho\left(F\left(z_j\right)\right) > M \qquad \forall \quad j > k.$$

For any $0 < R < 1$, let

$$A_R = \{z \;:\; R < |z| < 1\}.$$

By continuity of F_{M+2}, there exists $R < 1$ such that

$$\rho\left(F_{M+2}\left(A_R\right)\right) > M + 1.$$

Choose k such that $z_j \in A_R$ for all $j > k$. Then by Definition 2.6,

$$\rho\left(F_{i+1}\left(z\right)\right) - \rho\left(F_i\left(z\right)\right) > -\epsilon_i$$

for all $z \in \Delta$, so that

$$\rho\left(F\left(z\right)\right) - \rho\left(F_i\left(z\right)\right) > -\sum_{j=1}^{\infty} \epsilon_j > -1.$$

We conclude that

$$\rho\left(F\left(z_j\right)\right) > \rho\left(F_{M+2}\left(z_j\right)\right) - 1 > (M+1) - 1 > M$$

whenever $j > k$.

To complete the proof, note that any compact subset $\tilde{\Omega}$ of Ω is contained in a level set Ω_M for some M. The above analysis shows us that there exists an R such that $A_R \cap F^{-1}\left(\Omega_M\right) = \emptyset$. Hence $F^{-1}\left(\Omega_M\right) \subset\subset \Delta$ and we see that $F^{-1}\left(\tilde{\Omega}\right)$ is a compact subset of Δ. Therefore F is a proper analytic map from the open unit disk Δ into Ω.

CHAPTER 3

Handles for Strictly Pseudoconvex Domains

1. Overview

Our goal is to prove that it is possible to attach a lower-dimensional handle Σ to a strictly pseudoconvex domain D in such a way that the union $X = \overline{D} \cup \Sigma$ is holomorphically convex. In particular, we would like to show that for properly chosen handles, X has a Stein neighborhood basis. The techniques used in this proof will be elementary: nothing more than a little integration and some symmetry arguments.

We begin by finding an explicit Stein neighborhood basis for the most basic possible example: the unit ball $\mathbf{B}$ (or actually an ellipsoid $\mathbf{E}$) and a flat, totally real plane in $\mathbb{C}^2$. Note that $\mathbf{B}$ is strictly geometrically convex, *a forteriori* strictly pseudoconvex. Because the ball and plane have such simple geometry, it is possible to calculate exact conditions for neighborhoods Ω of X to be pseudoconvex. Once we obtain neighborhoods satisfying these conditions, they will provide the needed Stein neighborhood basis.

After finding a Stein neighborhood basis in this special case, we use some tricks to extend our results to strictly pseudoconvex domains. In essence, we will reduce the general case to the hyperbolic case, and the hyperbolic case to an adaptation of the construction for the ellipsoid.

In each of these reductions, as well as the construction of the Stein neighborhood basis for the ellipsoid and flat plane, the following well-known lemma will be useful:

Lemma 1.1. *Let ϕ_1 and ϕ_2 be smooth (strictly) plurisubharmonic functions defined on an open set $U \subset \mathbb{C}^n$, with ($d\phi_1 \neq d\phi_2$ on) $K = \{z \,:\, \phi_1(z) = \phi_2(z)\}$. Then there exists a smooth (strictly) plurisubharmonic function ψ on U such that*

- $\psi \geq max(\phi_1, \phi_2)$.
- $\psi = max(\phi_1, \phi_2)$ *outside a small neighborhood of K.*

- *$\psi - max(\phi_1, \phi_2)$ is arbitrarily small.*

For our purposes, Lemma 1.1 says that a domain D arising from the intersection of two strictly pseudoconvex domains D_1 and D_2 is strictly pseudoconvex in the sense that it can be approximated to arbitrary accuracy by a smooth strictly pseudoconvex domain contained within D. To see this, consider ϕ_1, ϕ_2 and ψ to be defining functions for the respective domains D_1, D_2 and D. Note that the lemma is a *local* result, so that the domain D is strictly pseudoconvex so long as it is *locally* a transverse intersection of strictly pseudoconvex domains.

Lemma 1.1 is a smoothing property of plurisubharmonic functions. For a proof see [**10**].

2. The ellipsoid E and the x_1-x_2 plane Σ_P

2.1. Preliminaries. We begin by defining precisely what we mean by an ellipsoid and the flat handle to which it will be attached. Let $z = (x_1 + iy_1, x_2 + iy_2)$ and define

$$\phi_{\mathbf{E}} = |x|^2 + c|y|^2 - 1 < 0 \tag{2.1}$$

for a constant c, so that our ellipsoid is

$$\mathbf{E} = \{z \ : \ \phi_{\mathbf{E}}(z) < 0\}.$$

Our flat handle must lie in a particular direction in order to attach properly to the ellipsoid. We let

$$\Sigma_P = \{z \ : \ y_1 = y_2 = 0\}.$$

so that the domain for which we want to find a Stein neighborhood basis is

$$X = \overline{\mathbf{E}} \cup \Sigma_P.$$

Finally, we let the ball $\mathbf{B}$ have the defining function

$$\phi_{\mathbf{B}}(z) = |z|^2 - 1.$$

First we will examine the structure of a (not necessarily Stein) neighborhood basis for the set X, and then we will try to find such a basis that is Stein. Our technique will involve finding a function that satisfies a certain differential inequality. For purposes of illustration, we will begin with a brief analysis of the case $\mathbf{E} = \mathbf{B}$, which will point the way to solving the problem for all $\mathbf{E}$.

Since X is cylindrically symmetric with respect to x_1 and x_2, and also with respect to y_1 and y_2, we consider a defining function ϕ for a neighborhood basis element Ω of the form

$$\phi(z) = |y|^2 - \chi\left(|x|^2\right) \tag{2.2}$$

where $\chi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, and let $\Omega = \{z \ : \ \phi(z) < 0\}$. If a neighborhood basis element Ω is defined by such a ϕ, then it will have the same cylindrical symmetry as X.

REMARK 2.1. If $\chi(t) = (1-t)/c$ then $\phi = \phi_{\mathbf{E}}$, so that Ω is just the ellipsoid. If instead $\chi(t) = $ Const., then Ω is a neighborhood of Σ_P. We want to choose χ to *interpolate* between these possibilities while keeping the Levi form of ϕ strictly positive in complex tangent directions. By doing so, we ensure that Ω remains strictly pseudoconvex.

We can compute the Levi form $\sum\limits_{j,k=1}^{n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k} \lambda_j \overline{\lambda}_k$ of ϕ and the complex tangents $\lambda = \left(\frac{\partial \phi}{\partial z_2}, -\frac{\partial \phi}{\partial z_1}\right)$ to the boundary $\partial\Omega = \{z \ : \ \phi(z) = 0\}$ *explicitly* in terms of derivatives of χ. Hence it is possible to write down the Levi condition for the set Ω to be strictly pseudoconvex explicitly in terms of derivatives of χ. See Appendix A for details of these computations. We find that the Levi form for *any* cylindrically symmetric Ω with a defining function as in Equation (2.2) is

$$\mathcal{L}_\phi(z, \lambda) = \frac{1}{2}\left(|y|^2 + |x|^2 (\chi')^2\right)(1 - \chi') - (x_1 y_2 - x_2 y_1)^2 \chi''. \tag{2.3}$$

We can bound the last term in Equation (2.3) from below to obtain a more symmetric estimate

$$\mathcal{L}_\phi(z, \lambda) > \frac{1}{2}\left(|y|^2 + |x|^2 (\chi')^2\right)(1 - \chi') - |x|^2 |y|^2 \chi''.$$

Using the fact that the Levi condition needs to hold only for z on the boundary $\partial\Omega$, where $\phi(z) = 0$ and hence $|y|^2 = \chi$, we substitute χ for $|y|^2$. We have reduced the problem of finding a neighborhood basis element that is Stein to the problem of locating a χ satisfying the following five conditions:

(i): $\chi(t) \geq (1-t)/c$ for $t < 1$, $\chi(t) > 0$ for $t \in [1, \infty)$.

(ii): $\chi'(t) = -1/c$ for $t = 1$.

(iii): $\chi''(t) \geq 0$ for all t.

(iv): $\chi(t) \equiv \delta_1$ for some constant δ_1 and all $t > 1 + a$ for a positive a.

(v): $\mathcal{L}_\chi = \left(\chi + |x|^2 (\chi')^2\right)(1 - \chi') - 2|x|^2 \chi\chi'' > 0$.

2.2. A First Try. Conditions **(i)** and **(ii)** arise from Remark 2.1. They imply that we are expecting χ' to interpolate between $\chi' = -1/c$ and $\chi' = 0$. It is therefore reasonable to assume (initially, at least) that $\chi' \leq 0$ in our region of interest. Also, the differential inequality **(v)** for $\mathcal{L}\chi$ can be written

$$\mathcal{L}\chi = |x|^2 \left((\chi')^2 - 2\chi\chi''\right) + \chi(1 - \chi') - \chi'\left(|x|^2 (\chi')^2\right) > 0. \tag{2.4}$$

Under the assumption $\chi' \leq 0$, the second two terms in Equation (2.4) are

$$\chi(1 - \chi') > 0$$

and

$$-\chi'\left(|x|^2 (\chi')^2\right) \geq 0$$

since χ is positive. We therefore need only to make the first term $|x|^2\left((\chi')^2 - 2\chi\chi''\right)$ positive, so we consider first the differential equation

$$(\chi')^2 - 2\chi\chi'' = 0 \tag{2.5}$$

or

$$\chi'' = \frac{(\chi')^2}{2\chi}. \tag{2.6}$$

Equation (2.6) can be solved by logarithmic integration to find that

$$\chi(t) = (C_1 - C_2 t)^2$$

for some arbitrary constants C_1 and C_2. For simplicity, we let $c = 1$ and take initial conditions

$$\begin{aligned} \chi(1) &= \epsilon \\ \chi'(1) &= -1 \end{aligned} \tag{2.7}$$

corresponding to the unit ball. We obtain the parabolic solution

$$\chi_0(t) = \frac{1}{4\epsilon}(1 + 2\epsilon - t)^2. \tag{2.8}$$

This χ_0 clearly satisfies Condition **(v)** on $[1, 1 + 2\epsilon)$ since $\chi_0' < 0$ in this region. However, at $t = |x|^2 = 1 + 2\epsilon$, we have $\chi_0 = \chi_0' = 0$ so that χ_0 fails Condition **(i)** above. That is, since χ_0 is not strictly positive it fails to provide an open neighborhood of the plane.

The obvious remedy to this problem is to "lift" χ_0 off the plane by considering the function $\chi_\delta(t) = \chi(t) + \delta$ for some small $\delta > 0$. Note that $\chi_\delta' = \chi_0'$ and $\chi_\delta'' = \chi_0''$. We must exercise caution, however, because this "lift" alters the Levi form of ϕ. Computing the new Levi form, we run into a new difficulty: $\chi'' \equiv 1/2\epsilon$, so when $t = 1 + 2\epsilon$, we see that Condition **(v)** has

$$\begin{aligned}\mathcal{L}_{\chi_\delta} &= \delta - 2(1+2\epsilon)(\delta)\chi_0'' \\ &< 0.\end{aligned}$$

We can analyze this problem in somewhat greater generality. Whenever $\chi_0' = 0$ we can write Equation (2.4) as

$$\begin{aligned}\mathcal{L}\chi = |x|^2\left(-2\chi\chi''\right) + \chi &= \chi\left(1 - |x|^2\chi''\right) \\ &> 0.\end{aligned}$$

From this, it is clear that we need to find a χ such that χ'' is smaller than 1 when $\chi' = 0$. No mere parabolic solution will suffice, but we can consider solutions of slightly higher powers than 2. As we will see, they provide the χ we need.

2.3. The Solution. Begin by writing the differential inequality in Condition **(v)** for $\mathcal{L}\chi$ as

$$\chi + |x|^2\left((\chi')^2 - 2\chi\chi''\right) - \chi\left(\chi'\right) - \chi'\left(|x|^2\left(\chi'\right)^2\right) > 0.$$

If we consider the region where $t = |x|^2 < 2$, then $\mathcal{L}\chi$ is positive if $\chi' \leq 0$ and the sum of the first two terms is positive:

$$\begin{aligned}& \chi + |x|^2\left((\chi')^2 - 2\chi\chi''\right) > 0 \\ \iff \quad & \frac{\chi}{|x|^2} + (\chi')^2 - 2\chi\chi'' > \frac{\chi}{2} + (\chi')^2 - 2\chi\chi'' > 0 \\ \iff \quad & \frac{\chi}{2} + (\chi')^2 > 2\chi\chi''.\end{aligned}$$

That is, we want

$$\begin{aligned}\chi'' &< \frac{1}{4} + \frac{(\chi')^2}{2\chi} \\ &= \frac{1}{8} + \frac{1}{8} + \frac{(\chi')^2}{2\chi},\end{aligned}$$

or

$$\chi'' < \underbrace{\frac{1}{8} + \frac{(\chi')^2}{(2-\eta)\chi}}_{A} + \underbrace{\frac{1}{8} - \frac{\eta(\chi')^2}{2(2-\eta)\chi}}_{B}$$

for any small η we choose. Note that we have split the right hand side into two pieces

$$A = \frac{1}{8} + \frac{(\chi')^2}{(2-\eta)\chi}$$

and

$$B = \frac{1}{8} - \frac{\eta(\chi')^2}{2(2-\eta)\chi}.$$

We want $\chi'' < A + B$ or, equivalently, $A - \chi'' + B > 0$.

Let χ_η solve

$$\chi'' = \frac{(\chi')^2}{(2-\eta)\chi} \tag{2.9}$$

so that $\chi'' = A - \frac{1}{8} < A$. We use initial conditions from Conditions **(i)** and **(ii)** corresponding to the ellipsoid to solve this differential equation

$$\chi(1) = \epsilon$$

$$\chi'(1) = -1/c$$

and we integrate to find the exact form of χ_η:

$$\chi_\eta(t) = |C_1 - C_2 t|^{\frac{2-\eta}{1-\eta}} \tag{2.10}$$

with

$$C_1 = \epsilon^{\frac{1-\eta}{2-\eta}} + \frac{1}{c}\left(\frac{1-\eta}{2-\eta}\right)\epsilon^{\frac{-1}{2-\eta}}$$

$$C_2 = \frac{1}{c}\left(\frac{1-\eta}{2-\eta}\right)\epsilon^{\frac{-1}{2-\eta}}.$$

We then obtain

$$\chi_\eta' = -\frac{2-\eta}{1-\eta}C_2|C_1 - C_2 t|^{\frac{1}{1-\eta}}$$

and

$$\chi_\eta'' = \frac{2-\eta}{1-\eta}\frac{1}{1-\eta}C_2^2|C_1 - C_2 t|^{\frac{\eta}{1-\eta}}.$$

Of special interest to us is the fact that when $\eta > 0$, the solution χ_η is a power of order $\approx 2 + \eta \neq 2$, so that the second derivative χ_η'' is no longer constant. As a matter of fact, χ_η'' is a power of order $\approx \eta$, so the size of χ_η'' is small when χ_η and the first derivative χ_η' are small. This satisfies the requirement for a small χ_η'' noted above. Let us make this analysis precise:

Define $t_0 = C_1/C_2 \approx 1 + 2c\epsilon$ to be the minimum for χ_η, and note that

$$\chi_\eta(t_0) = \chi_\eta'(t_0) = \chi_\eta''(t_0) = 0.$$

χ_η'' increases with distance from t_0; on the interval $[1, t_0]$, we have an upper bound

$$\begin{aligned} \chi_\eta'' &\leq \frac{2-\eta}{(1-\eta)^2} C_2^2 |C_1 - C_2|^{\frac{\eta}{1-\eta}} \\ &= \frac{2-\eta}{(1-\eta)^2} C_2^2 \epsilon^{\frac{\eta}{2-\eta}} \\ &= \frac{1}{c^2\,(2-\eta)} \frac{1}{\epsilon}. \end{aligned}$$

Call this maximum

$$M_2 = \frac{1}{c^2\,(2-\eta)} \frac{1}{\epsilon}.$$

Then if we choose

$$\eta < \frac{\epsilon c^2}{2 + \epsilon c^2/2} \tag{2.11}$$

we will have that

$$\frac{1}{8} - \frac{\eta}{2}\chi_\eta'' \geq \frac{1}{8} - \frac{\eta}{2} M_2 > 0$$

which means that

$$\begin{aligned} B &= \frac{1}{8} - \frac{\eta}{2} \frac{(\chi_\eta')^2}{(2-\eta)\chi_\eta} \\ &= \frac{1}{8} - \frac{\eta}{2}\chi_\eta'' \\ &> 0 \qquad \text{for all } t \in [1, t_0] \end{aligned} \tag{2.12}$$

where we have used the fact that χ_η satisfies the differential equation (2.9). Thus the quantity B is positive, and we find that

$$\chi'' = A - \frac{1}{8} < A < A + B,$$

or equivalently

$$A - \chi'' + B > 0.$$

Condition **(v)** is therefore true when $t \in [1, t_0)$, so we can conclude that the set Ω defined by

$$\Omega = \{z \;:\; |y|^2 - \chi\left(|x|^2\right) < 0\}$$

is strictly pseudoconvex in the region $1 \leq |x|^2 < t_0$.

Notice, however, that χ_η shares the drawback of χ_0 (as in Equation (2.8)) in that it is not strictly positive – in particular, $\chi(t_0) = 0$. In order to satisfy Condition **(i)** and get an open neighborhood of the plane, we want to "lift" χ_η as we tried to do with χ_0. That is, we want to choose $\chi = \chi_\eta + \delta$ for some δ so small that our differential quantity $A - \chi'' + B$ will *stay* positive. The reason that we expect to be able to do so is that χ_η, unlike χ_0, has a second derivative which is zero when the first derivative is zero.

Note that for this new χ, $\chi' = \chi_\eta'$ and $\chi'' = \chi_\eta''$ so

$$\begin{aligned}
&1/(\chi_\eta + \delta) < 1/\chi_\eta \\
\Longrightarrow \frac{1}{8} - \frac{\eta}{2}\frac{(\chi')^2}{(2-\eta)\chi} &= \frac{1}{8} - \frac{\eta}{2}\frac{(\chi_\eta')^2}{(2-\eta)(\chi_\eta+\delta)} \\
&> \frac{1}{8} - \frac{\eta}{2}\frac{(\chi_\eta')^2}{(2-\eta)\chi_\eta} \\
&> 0
\end{aligned}$$

Therefore our estimate $B > 0$ in Equation (2.12) still holds for our new χ. We need only look for a δ so small that A satisfies

$$\begin{aligned}
\chi_\eta'' &< A \\
&= \frac{1}{8} + \frac{(\chi')^2}{(2-\eta)(\chi)} \\
&= \frac{1}{8} + \frac{(\chi_\eta')^2}{(2-\eta)(\chi_\eta+\delta)} \\
&= \frac{1}{8} + \underbrace{\frac{(\chi_\eta')^2}{(2-\eta)\chi_\eta}}_{\chi_\eta'' \text{ (by definition)}} \left(1 - \frac{\delta}{\chi_\eta+\delta}\right) \\
&= \frac{1}{8} + \chi_\eta'' - \chi_\eta''\left(\frac{\delta}{\chi_\eta+\delta}\right).
\end{aligned}$$

That is, we need

$$0 < \frac{1}{8} - \chi_\eta''\frac{\delta}{\chi_\eta+\delta}.$$

Now

$$\chi_\eta'' = \text{Const} \cdot |t_0 - t|^{\eta + o(\eta)}$$

so it increases with distance from t_0. As a matter of fact, we can compute that $\chi_\eta'' < 1/16$ whenever t is of distance less than

$$s = \left(\frac{2-\eta}{16}\right)^{\frac{1-\eta}{\eta}} \left(\frac{2-\eta}{1-\eta}\right) c^{\frac{2-\eta}{\eta}} \epsilon^{\frac{1}{\eta}} \tag{2.13}$$

from t_0. We also have

$$\chi = \text{Const} \cdot |t_0 - t|^{2+\eta+o(\eta)} + \delta \tag{2.14}$$

so that on the interval $[1, t_0 - s]$, χ_η takes on its minimum

$$M_0 := C_2^{\frac{2-\eta}{1-\eta}} s^{\frac{2-\eta}{1-\eta}}$$

at the endpoint $t = t_0 - s$.

Choose

$$\delta < \frac{M_0}{16M_2 - 1} \tag{2.15}$$

so that

$$\frac{\delta}{M_0 + \delta} < \frac{1}{16M_2}.$$

Then on the interval $[1, t_0 - s]$,

$$\begin{aligned} \frac{1}{8} - \chi_\eta'' \frac{\delta}{\chi_\eta + \delta} &\geq \frac{1}{8} - M_2 \frac{\delta}{\chi_\eta + \delta} \\ &> \frac{1}{8} - M_2 \frac{\delta}{M_0 + \delta} \\ &> \frac{1}{8} - \frac{1}{16} \\ &> 0 \end{aligned}$$

and on the interval $[t_0 - s, t_0]$,

$$\begin{aligned} \frac{1}{8} - \chi_\eta'' \frac{\delta}{\chi_\eta + \delta} &\geq \frac{1}{8} - \frac{1}{16} \frac{\delta}{\chi_\eta + \delta} \\ &\geq \frac{1}{8} - \frac{1}{16}(1) \\ &> 0. \end{aligned}$$

Hence, by our choices of η and δ, we obtain $A - \chi'' + B > 0$ when $t \in [1, t_0]$. Therefore the Levi form of $\phi = |y|^2 - (\chi_\eta(|x|^2) + \delta)$ is strictly positive for $|x|^2$ in the closed interval $[1, t_0]$. We also know that the function $\phi = \phi_\chi$ will give an open neighborhood $\widetilde{\Omega}_2$ of the ellipsoid and plane.

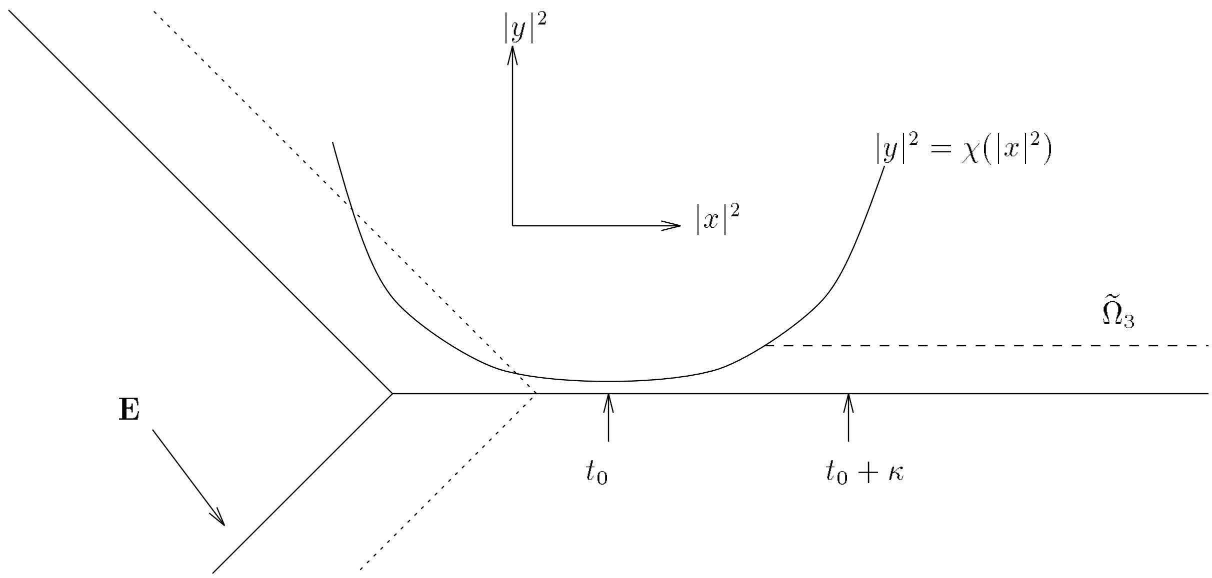

FIGURE 3.1. The neighborhood Ω of $\mathbf{B}$ and Σ_P

By compactness of the interval $[1, t_0]$, there exists some interval $[1, t_0 + \kappa]$ for a $\kappa > 0$ on which the Levi form of ϕ remains strictly positive. Choose a

$$\delta_1 \in (\chi(t_0), \chi(t_0 + \kappa)).$$

Equation (2.14) implies that $\delta_1 > \delta$. Geometrically, δ_1 is the height of χ in a region past the cylinder $|x|^2 = t_0$ where the Levi form of ϕ is still remaining strictly positive (see Figure 3.1). We will use this δ_1 as a thickness for a neighborhood $\widetilde{\Omega}_3$ of the plane Σ_P, and this choice will allow us to use Lemma 1.1 to link $\widetilde{\Omega}_3$ to $\widetilde{\Omega}_2$.

REMARK 2.2. One can, of course, explicitly compute κ. If η and δ are chosen to be, say, half the size of their upper bounds given above in Equations (2.11) and (2.15), then it is elementary to compute that $\kappa = O(\epsilon^2)$.

We are now ready to use Lemma 1.1 to complete our construction. We combine the neighborhood $\widetilde{\Omega}_2$ above with neighborhoods $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_3$ of the ellipsoid and the plane respectively to provide a Stein neighborhood basis element Ω.

When we take the domain

$$\widetilde{\Omega}_1 = \mathbf{E}_{1+2\epsilon} = \{z \ : \ |x|^2 + c|y|^2 < 1 + 2\epsilon\}$$

and the domain $\widetilde{\Omega}_2$ defined as above by χ, they intersect on four polycircles on $\mathbf{E}_{1+2\epsilon}$, specified by $|x|^2 = a_1$, $|y|^2 = 1 - a_1$ and $|x|^2 = a_2 > a_1$, $|y|^2 = 1 - a_2$ for some $a_1, a_2 \in \mathbb{R}^+$.

Similarly, the neighborhood of Σ_P given by $\widetilde{\Omega}_3 = \{z \ : \ |y|^2 - \delta_1 < 0\}$ intersects $\widetilde{\Omega}_2$ in four more polycircles, specified by $|x|^2 = a_3$, $|y|^2 = \delta_1$ and $|x|^2 = a_4 > a_3$, $|y|^2 = \delta_1$ where

$$\left\{t \ : \ t \in \chi_\eta^{-1}(\delta_1 - \delta)\right\} = \{a_3, a_4\}.$$

Define a neighborhood $\widetilde{\Omega}$ of X by

$$\widetilde{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

where

$$\begin{aligned}
\Omega_1 &= \{z \ : \qquad\quad |x|^2 \le a_1 &&\text{and } z \in \widetilde{\Omega}_1 = \mathbf{E}_{1+2\epsilon}\} \\
\Omega_2 &= \{z \ : \ a_1 \le |x|^2 \le a_4 &&\text{and } z \in \Omega\} \\
\Omega_3 &= \{z \ : \ a_4 \le |x|^2 &&\text{and } |y|^2 < \delta_1\}.
\end{aligned}$$

That is, if we let

$$\tilde{\rho}(z) = \begin{cases} c|y|^2 + |x|^2 - (1+2\epsilon) & \text{if } \qquad |x|^2 \le a_1 \\ |y|^2 - \chi_\eta\left(|x|^2\right) - \delta & \text{if } a_1 \le |x|^2 \le a_4 \\ |y|^2 - \delta_1 & \text{if } a_4 \le |x|^2 \end{cases} \tag{2.16}$$

then $\tilde{\rho}$ is a defining function for $\widetilde{\Omega}$. We can use Lemma 1.1 on $\tilde{\rho}$ to create a new domain Ω which is a smooth neighborhood of the ellipsoid $\mathbf{E}$ and the plane Σ_P and which is strictly pseudoconvex everywhere along its boundary. The distance from $\partial\Omega$ to $\overline{\mathbf{E}} \cup \Sigma_P$ depends on ϵ, so Ω forms an element in the Stein neighborhood basis we seek:

THEOREM 2.3. *There exists a collection of smoothly bounded, strictly pseudo-convex sets* $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_j \supset \cdots$ *such that*

$$\overline{\mathbf{E}} \cup \Sigma_P = \bigcap_{j \in \mathbb{N}} \Omega_j.$$

2.4. Extensions. The final step above depended only on our ability to ensure that the ball $\mathbf{B}_{1+2\epsilon}$ and the neighborhood $\widetilde{\Omega}_3$ of the plane intersected $\widetilde{\Omega}_2$ transversally. Thus, we have a corollary to the construction above:

COROLLARY 2.4. *If Σ is a smooth totally real hyperplane in $\mathbb{C}^2$ intersecting $\overline{\mathbf{E}}$ on the circle $|x|^2 = 1$ such that for points $z \in \Sigma$ near this circle we have*

$$|y|^2 \leq e^{\log \zeta / \zeta} \qquad \text{where } \zeta = \frac{c^2}{20}\left(|x|^2 - 1\right)$$

then there exists a smooth Stein neighborhood basis for the set $X = \overline{\mathbf{E}} \cup \Sigma$.

PROOF. This follows directly from the formula for δ in Equation (2.15) and the formula for s in Equation (2.13). Note that these formulas make it immediately clear that $\delta \approx \epsilon^{1/\epsilon}$. $\square$

Similarly, it is possible to begin with a small perturbation of the ellipsoid $\mathbf{E}$ and still find a Stein neigborhood basis for X.

We now consider the problem of extending our results to higher dimensions.

DEFINITION 2.5. A *complex k-plane* L in $\mathbb{C}^n$ is a complex affine image of $\mathbb{C}^k$. That is, there exists a linear map $f : \mathbb{C}^k \to L \subset \mathbb{C}^n$ such that

$$f(z_1, \ldots, z_k) = p + \sum_{j=1}^{k} a_j z_j$$

for some constants a_j and p in $\mathbb{C}^n$.

The following theorem is part of the general folklore:

THEOREM 2.6. *A smoothly bounded domain $\Omega \subset\subset \mathbb{C}^n$ with defining function ρ is strictly pseudoconvex if and only if for every boundary point $p \in \partial\Omega$ and every complex 2-plane L through p such that $d\rho \in T(L) = T_{\mathbb{C}}(L)$, the domain $\Omega \cap L$ is strictly pseudoconvex when considered as a subset of $\mathbb{C}^2$, i.e. if and only if $f^{-1}(\Omega \cap L)$ is strictly pseudoconvex in $\mathbb{C}^2$.*

PROOF. CASE 1. ($\Rightarrow$) Since Ω is already pseudoconvex, with a complex tangent space to its boundary $\partial\Omega$ of $n-1$ dimensions, L can contain only one complex tangent vector λ to $\partial\Omega$. Assume without loss of generality that $\lambda = (1, 0, \ldots, 0)$. Then the Levi form

$$\begin{aligned} \mathcal{L}_\rho(p, \lambda) &= \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} \lambda_j \overline{\lambda}_k \\ &= \frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_1} \lambda_1 \overline{\lambda}_1 \\ &> 0 \end{aligned} \tag{2.17}$$

by virtue of the strict plurisubharmonicity of ρ.

CASE 2. ($\Leftarrow$) Let λ be a complex tangent vector to $\partial\Omega$ at p, and let $\nu = d\rho|_p$. Let L be the complex 2-plane defined by

$$f(z_1, z_2) = p + \lambda z_1 + \nu z_2.$$

and define $\tilde{\rho}$ to be the restriction of ρ to L. Using a translation and a complex rotation, we can arrange L and Ω so that $\lambda = (1, 0, \dots)$ and $\nu = (0, 1, 0, \dots)$. Then by assumption $\mathcal{L}_{\tilde{\rho}}(p, \lambda) > 0$, and

$$\frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_1} = \frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z}_1}.$$

Now

$$\begin{aligned} \mathcal{L}_\rho(p, \lambda) &= \frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_1} \lambda_1 \overline{\lambda}_1 \\ &= \frac{\partial^2 \tilde{\rho}}{\partial z_1 \partial \overline{z}_1} \lambda_1 \overline{\lambda}_1 \\ &> 0. \end{aligned}$$

Thus $\mathcal{L}_\rho(p, \lambda) > 0$ for all complex tangents λ to $\partial\Omega$, and Ω is strictly pseudoconvex.

$\square$

THEOREM 2.7. *If $\Omega \subset \mathbb{C}^n$ is a set with a $\mathcal{C}^2$ defining function $\phi(z) = |y|^2 - \chi(|x|^2)$ with the property that the sets*

$$\Omega_{jk} := \{z \in \Omega \ : \ z_l = 0 \ \textit{if}\ l \neq j, k\}$$

are strictly pseudoconvex when considered as subsets of $\mathbb{C}^2$, then Ω is strictly pseudoconvex.

REMARK 2.8. This theorem is not an immediate consequence of Theorem 2.6, because the sets Ω_{jk} do not constitute all possible complex 2-planes through the boundary $\partial\Omega$ (and containing its normal vector). However, Theorem 2.6 does point the way to a proof:

PROOF. For a point $p = (s_1 + it_1, \dots, s_n + it_n) \in \partial\Omega$, let $M \in O[n, \mathbb{R}]$ be a real orthogonal $n \times n$ rotation matrix such that

$$M \cdot \mathbf{s} = (x, 0, \dots, 0)$$

for some $x \in \mathbb{R}^+$. Since $O[n, \mathbb{R}]$ has $n-1$ degrees of freedom, we can also specify that

$$M \cdot \mathbf{t} = (y_1, y_2, 0, \ldots, 0)$$

for some $y_1, y_2 \in \mathbb{R}$. Because M is a rotation matrix, it is norm-preserving, so $x = |\mathbf{s}|$ and $y_1^2 + y_2^2 = |\mathbf{t}|^2$.

Now consider M as a rotation matrix in $SU[n, \mathbb{C}]$ that happens to have all real entries. $M \cdot \mathbf{z}$ is a complex rotation of coordinates, so M represents a holomorphic (actually a complex linear) change of coordinates which we will denote by $h(z)$. Note

$$M \cdot (\mathbf{s} + i\mathbf{t}) = M \cdot \mathbf{s} + iM \cdot \mathbf{t},$$

so h preserves both $|x|$ and $|y|$. Thus by the cylindrical symmetry of Ω, we find that $h(p) \in \Omega \Leftrightarrow p \in \Omega$, i.e. Ω is invariant under h. If λ is a complex tangent vector to $\partial\Omega$, then so is $h(\lambda)$.

The above argument allows us to reduce the question of pseudoconvexity at any boundary point of Ω to that of pseudoconvexity at a point of the type

$$p = (x + iy_1, iy_2, 0, \ldots, 0).$$

Thus we need:

LEMMA 2.9. *Given a domain Ω and a defining function ϕ as in Theorem 2.7, as well as a point $p \in \Omega$ of the form $p = (x_1 + iy_1, iy_2, 0, \ldots, 0)$ and a complex tangent vector ν to $\partial\Omega$ at p, we can conclude that $\mathcal{L}(p, \nu) > 0$.*

PROOF. Computing the partial derivatives of ϕ at p, we find that

$$\left.\frac{\partial \phi}{\partial z_j}\right|_p = -iy_j - x_j\chi' = 0 \tag{2.18}$$

when $j > 2$ and

$$\left.\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}\right|_p = \frac{1}{2}\delta_{jk} - x_j x_k \chi'' = \frac{1}{2}\delta_{jk} \tag{2.19}$$

when $j > 1$ or $k > 1$. Equation (2.18) implies that

$$\nu = (\nu_1, \nu_2, \ldots, \nu_n)$$

where (ν_1, ν_2) is a complex tangent vector to Ω_{12}, and ν_j is arbitrary for $j > 2$.

From Equation (2.19) we obtain

$$\begin{aligned}
\mathcal{L}(p,\nu) &= \sum_{j,k=1}^{n} \frac{\partial^2\phi}{\partial z_j \partial \overline{z}_k} \nu_j \overline{\nu}_k \\
&= \sum_{j,k=1}^{2} \frac{\partial^2\phi}{\partial z_j \partial \overline{z}_k} \nu_j \overline{\nu}_k + \sum_{j=3}^{n} \frac{\partial^2\phi}{\partial z_j \partial \overline{z}_j} |\nu_j|^2 \\
&= \mathcal{L}^*(p,(\nu_1,\nu_2)) + \sum_{j=3}^{n} \frac{1}{2}|\nu_j|^2
\end{aligned}$$

where $\mathcal{L}^*$ denotes the Levi form taken in the complex 2-plane slice corresponding to the z_1 and z_j directions. By assumption this term involving $\mathcal{L}^*$ is positive, so we obtain $\mathcal{L}(p,\nu) > 0$. □

According to Lemma 2.9, we see that $\mathcal{L}_\phi(h(p),\nu) > 0$ for any complex tangent vector ν to $\partial\Omega$ at $h(p)$. Since $\partial\Omega$ is invariant under h, given any complex tangent vector λ to $\partial\Omega$ at p, we have that $\nu = h(\lambda)$ is complex tangent at $h(p)$. Section 3.1 of [**11**] shows that plurisubharmonicity is invariant under holomorphic maps, so

$$\begin{aligned}
\mathcal{L}_\phi(h(p),\nu) = \mathcal{L}_\phi(h(p),h(\lambda)) > 0 \\
\Longrightarrow \qquad \mathcal{L}_\phi(p,\lambda) > 0
\end{aligned}$$

for all p and λ. □

At last we are able to extend our results to higher dimensions:

Corollary 2.10. *For any $n \geq 2$, there exists a Stein neighborhood basis for the set $\overline{\mathbf{E}} \cup \Sigma_P \subset \mathbb{C}^n$ where*

$$\mathbf{E} := \left\{ z \in \mathbb{C}^n \ : \ |x|^2 + c|y|^2 \leq 1 \right\},$$

and

$$\Sigma_P := \left\{ z \in \mathbb{C}^n \ : \ |y|^2 = 0 \right\}$$

with $z = (x_1 + iy_1, \ldots, x_n + iy_n)$ and $|y|^2 = y_1^2 + \cdots + y_n^2$.

Proof. Given $\epsilon > 0$, let $\rho(z)$ be defined as in equation (2.16) with $|x|^2$ and $|y|^2$ defined now in $\mathbb{C}^n$. Then the domain Ω defined by ρ fits the hypotheses of Theorem 2.7, and hence is strictly pseudoconvex. According to Lemma 1.1, Ω can be smoothed to a smooth Stein neighborhood basis element for $\overline{\mathbf{E}} \cup \Sigma_P$. □

3. Hyperbolic domains of the form $|x|^2 - c|y|^2 - 1$

We are ready to extend our results to the case of domains which are not elliptical. For now, however, we retain the cylindrical symmetry and consider hyperbolic domains Ω with defining functions ρ of the form $\rho(z) = |x|^2 - c|y|^2 - 1$. Since we are interested in the strictly pseudoconvex case, we restrict our attention to the case $c < 1$.

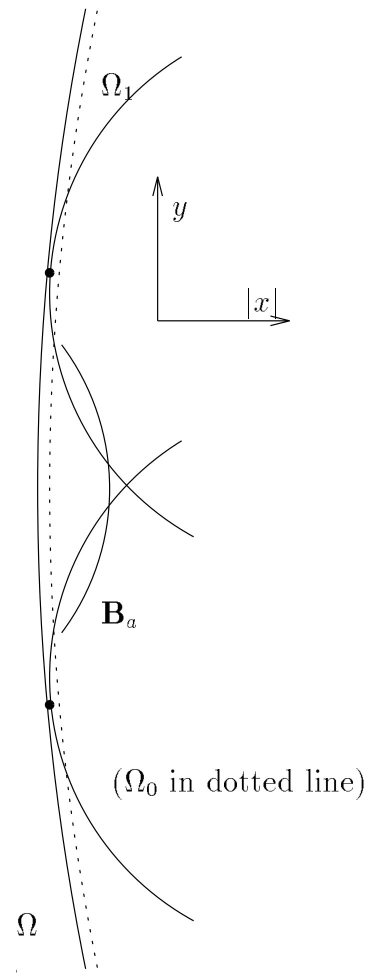

FIGURE 3.2. The construction of a neighborhood for Ω

Conceptually, our method is relatively simple: we bound Ω behind two hyperbolic domains Ω_1 of slightly higher curvature – one translated up a bit, and the other translated down. Locally, where these new domains meet, their intersection forms a strictly convex domain. This domain can therefore be locally bounded by a ball $\mathbf{B}_a$, for which we already know how to construct a Stein neighborhood basis element. See Figure 3.2 for a drawing of this construction.

We begin by specifying the translated hyperbolic domain Ω_1 with more precision. As we can see from the drawing, we want Ω_1 to just about intersect Ω tangentially, so that it will intersect a neighborhood Ω_0 of Ω transverally. We therefore consider a defining function for Ω_1 of the form

$$\rho_1(z) = |x|^2 - d(|y| - \eta)^2 - (1+\delta).$$

The constant d introduces the higher curvature, while the constants η and δ provide the necessary translation. Note that the Levi form of $|y|$ satisfies

$$\begin{aligned}\mathcal{L}_{|y|^2}(z,\lambda) &= \sum_{j,k=1}^{n} \frac{1}{4|y|}\left(\delta_{jk} - \frac{y_j}{|y|}\frac{y_k}{|y|}\lambda_j\overline{\lambda}_k\right) \\ &= \frac{|\lambda|^2}{4|y|} - \frac{1}{4|y|^3}|y\cdot\lambda|^2 \\ &\geq \frac{|\lambda|^2}{4|y|} - \frac{1}{4|y|^3}|y|^2|\lambda|^2 \\ &= 0\end{aligned}$$

so that

$$\mathcal{L}_{\rho_1}(z,\lambda) \geq \frac{1-d}{2}|\lambda|^2 > 0$$

and therefore Ω_1 is strictly pseudoconvex so long as $d < 1$. This leads us to choose

$$d = \frac{c+1}{2} \tag{3.1}$$

$$\delta < \frac{c^3(1-c)^3}{2} \tag{3.2}$$

and

$$\eta = \sqrt{\frac{\delta(1-c)}{c(1+c)}}. \tag{3.3}$$

That is, d is halfway between the constant c and 1, while δ is an arbitrarily small constant bounded above by a constant depending on c. We compute

$$\rho(z) - \rho_1(z) = (d-c)|y|^2 - 2\eta d|y| + \left(d\eta^2 + \delta\right) \tag{3.4}$$

$$= (d-c)\left(|y| - \eta\frac{1+c}{1-c}\right)^2 \tag{3.5}$$

$$\geq 0. \tag{3.6}$$

It is now clear that the constant η was chosen in such a way that $\partial\Omega_1$ intersects $\partial\Omega$ tangentially. We compute to find out exactly where $\partial\Omega_1$ intersects $\partial\Omega$:

$$\rho(z) = \rho_1(z) = 0 \tag{3.7}$$

$$\begin{aligned} &\implies \quad |y|^2 = \left(\eta\frac{1+c}{1-c}\right)^2 \quad \text{and} \quad |x|^2 = 1 + c|y|^2 \\ &\implies \quad |y|^2 = \frac{\delta(1+c)}{c(1-c)} \qquad \text{and} \quad |x|^2 = 1 + \delta\frac{1+c}{1-c}. \end{aligned} \tag{3.8}$$

Since we actually want Ω_1 to be an open neighborhood of $\overline{\Omega}$, we make one final adjustment; we replace η by a slighty smaller η and redefine ρ_1 and Ω_1 accordingly. Then Ω_1 will be an open neighborhood of $\overline{\Omega}$ and will tranversally intersect a level neighborhood Ω_0 of $\overline{\Omega}$ defined by

$$\Omega_0 = \{z \ : \ \rho(z) < \gamma\}$$

for some constant $\gamma > 0$.

Now we are ready to show that it is possible to cut Ω_1 off with a ball (see Figure 3.2 again). If we consider the region where $|y|$ is so small that $|y| \ll \eta d$, then we see that

$$\rho_1(z) = |x|^2 - (1 + \delta + \eta^2 d) + 2\eta d|y| + O(|y|^2).$$

Since $|x| \approx 1$, this is approximately linear in $|y|$. Choose a small $\epsilon \ll \eta d$, and let

$$a = 1 + \delta + \eta^2 d - \epsilon$$

with

$$\mathbf{B}_a = \left\{z \ : \ |z|^2 < a\right\}.$$

and

$$\rho_a(z) = |x|^2 + |y|^2 - a.$$

Then for $|y| < \epsilon$, we see that

$$\begin{aligned} \rho_1(z) - \rho_a(z) &= 2\eta d|y| - \left(|y|^2 + \epsilon\right) + O(|y|^2) \\ &= 2\eta d|y| - \epsilon + O(|y|^2) \\ &< |y| - \epsilon \\ &< 0. \end{aligned}$$

At the same time, since

$$\rho_1(z) \geq |x|^2 + |y|^2 - (1 + \delta + \eta^2 d).$$

for small $|y|$, we see that the intersection of $\mathbf{B}_a$ and Ω_1 must be transverse.

We have now constructed a neighborhood for $\overline{\Omega}$ which is strictly pseudoconvex, and which looks like the ball $\mathbf{B}_a$ for small $|y|$. From Corollary 2.10 in Section 2, we know that we can construct a cylindrically symmetric Stein neighborhood basis for the set $\overline{\mathbf{B}}_a \cup \Sigma_P$. Choose a neighborhood basis element Ω_2 from this Stein neighborhood basis which is so close to the set $\overline{\mathbf{B}}_a \cup \Sigma_P$ that it, too, has a transverse intersection with Ω_1. Let $\rho_2(z)$ be the defining function for Ω_2.

Now we want to see that by taking appropriate local intersections of Ω_1, Ω_2 and Ω_3, we can construct a Stein neighborhood basis element for $\overline{\Omega} \cup \Sigma_P$. We do so by creating an appropriate defining function.

Since both Ω_1 and Ω_2 are cylindrically symmetric, there exist constants $a_3, b_3 > 0$ such that

$$\partial\Omega_1 \cap \partial\Omega_2 = \left\{ z \ : \ |x|^2 = a_3, |y|^2 = b_3 \right\}.$$

Similarly, Equation (3.8) implies that there exist more constants

$$a_1 > 1 + \delta \frac{1+c}{1-c} > a_2 > 0$$

and

$$b_1 > \frac{\delta(1+c)}{c(1-c)} > b_2 > 0$$

such that

$$\partial\Omega_0 \cap \partial\Omega_1 = \left\{ z \ : \ |x|^2 = a_1, |y|^2 = b_1 \right\} \cup \left\{ z \ : \ |x|^2 = a_2, |y|^2 = b_2 \right\}.$$

Furthermore, our choice of ϵ ensures that $b_2 > b_3$. We therefore define

$$\tilde{\rho}(z) = \begin{cases} \rho_2(z) & \text{if} \qquad |y|^2 \leq b_3 \\ \rho_1(z) & \text{if } b_3 \leq |y|^2 \leq b_1 \\ \rho(z) - \gamma & \text{if } b_1 \leq |y|^2 \end{cases}$$

Finally, we use Lemma 1.1 to smooth $\tilde{\rho}$. We obtain a smoothly bounded strictly pseudoconvex neighborhood basis element whose distance from $\Omega \cup \Sigma_P$ decreases with the arbitrarily small constant δ chosen in Equation (3.2). We have therefore proven:

THEOREM 3.1. *If Ω is a hyperbolic domain with a defining function of the form*

$$\rho(z) = |x|^2 - c|y|^2 - 1$$

for some $0 \leq c < 1$, then there exists a Stein neighborhood basis for the set $\overline{\Omega} \cup \Sigma_P$

To state things a little more generally, we combine Theorem 3.1 with Corollary 2.10 to obtain

COROLLARY 3.2. *Given any cylindrically symmetrical set Ω with a defining function of the form*

$$\rho(z) = |x|^2 - c|y|^2 - 1$$

for some $c < 1$, there exists a Stein neighborhood basis for the set $\overline{\Omega} \cup \Sigma_P$.

4. The general hyperbolic case

DEFINITION 4.1. Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with a real analytic defining function $\rho(z)$. Let $D_\epsilon := \{z \in \mathbb{C}^n \ : \ \rho(z) < \epsilon\}$. A totally real hypersurface Σ, of real dimension n, is a *handle* for D if there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$, $S_\epsilon := \Sigma \cap \partial D_\epsilon$ is isomorphic to the n-sphere S^n, and if for points P along S_ϵ, $d\rho \in T(\Sigma)$ and $T(\Sigma)/d\rho \subset T_{\mathbb{C}}(\partial D_\epsilon)$.

We will use our results from the previous sections to construct a Stein neighborhood basis for the set $\overline{D} \cup \Sigma$, when Σ is a flat plane in $\mathbb{C}^n$ and $\partial D \cap \Sigma$ is a circle. First of all, note that there exists a complex linear change of coordinates taking Σ to the x_1-...-x_n plane Σ_P. We can therefore assume without loss of generality that $\Sigma = \Sigma_P$ and that $\partial D \cap \Sigma$ is the unit n-sphere in Σ_P.

Definition 4.1 and the flatness of our handle create certain restrictions on the form ρ may take. Define

$$x' = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

and

$$y' = \sqrt{|y_1|^2 + \cdots + |y_n|^2}.$$

A priori, we know that the sphere S_0 is in ∂D, so near S_0 we can write

$$\rho(z) = |x'|^2 - 1 + \sum_{j=1}^{n} y_j g^j(z) + \sum_{j \geq k=1}^{n} y_j y_k g^{jk}(z) + o\left(|y'|^2, |x'|^2 - 1\right)$$

where the functions g^j and g^{jk} are real analytic. However, the condition that $d\rho \in T(\Sigma)$ along S_ϵ requires that the second term satisfy

$$d\left(\sum_{j=1}^{n} y_j g^j(z)\right) \in T(\Sigma)$$

which is possible only if $g^j(z) = 0$ for all z. Hence

$$\rho(z) = |x'|^2 - 1 + \sum_{j,k=1}^{n} y_j y_k g^{jk}(z) + o\left(|y'|^2, |x'|^2 - 1\right). \tag{4.1}$$

From the fact that the Levi form of ρ is positive, we can glean information about the size of the g^{jk}. Namely on S_0,

$$\mathcal{L}(\rho, \lambda) = \frac{1}{2}|\lambda|^2 + \frac{1}{2}\sum_{j \geq k=1}^{n} g^{jk}(z)\lambda_j \overline{\lambda}_k > 0.$$

Hence if we substitute $\lambda = (y_1, \ldots, y_n)$, we find that

$$\sum_{j \geq k=1}^{n} g^{jk}(z) y_j y_k < |y'|^2. \tag{4.2}$$

Since (4.2) holds on the entirety of the compact set S_0, there exists a number $c < 1$ such that

$$\sum_{j \geq k=1}^{n} g^{jk}(z) y_j y_k < c|y'|^2$$

for all $z \in S_0$. Hence in a neighborhood of S_0,

$$\rho(z) < \rho_1(z) := |x'|^2 - c|y'|^2 - 1. \tag{4.3}$$

It is now easy to see how we must approach the proof of:

THEOREM 4.2. *Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with real analytic boundary ∂D, and let Σ be a flat (that is, planar) handle for D. Then there exists a Stein neighborhood basis for the set $\overline{D} \cup \Sigma$.*

PROOF. As before, we will use Lemma 1.1 to define an arbitrarily small basis element in such a neighborhood basis.

Assume without loss of generality that we have a defining function ρ for D of the form (4.1), and let ρ_1 be as in equation (4.3). A level set D_ϵ for a very small ϵ will intersect the set D_1 defined by ρ_1 transversally. Hence we use these two sets to construct a Stein neighborhood basis element for D. $\square$

5. The general strictly pseudoconvex case

We are finally ready to state and prove the most general case of our results on handles.

THEOREM 5.1. *Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with boundary ∂D, and let Σ be a real analytic handle for D. Then there exists a Stein neighborhood basis for the set $\overline{D} \cup \Sigma$.*

PROOF. Taking ϵ from Definition 4.1 (the definition of a handle), we define the compact set

$$K = \Sigma \cap \overline{D}_\epsilon \setminus D.$$

Let a real analytic function $f : K \to \mathbb{C}^n$ take K to the totally real x_1-...-x_n plane, in such a way that

$$f(\partial D \cap K) = \left\{ z \ : \ |x|^2 = 1 \right\}$$

and

$$f(K \setminus \partial D) \subset \left\{ z \ : \ |x|^2 > 1 \right\}.$$

Then since f has a holomorphic extension locally for each $z_0 \in K$, there exists a holomorphic extension F of f in a neighborhood U of K (see, for example, the partion of unity argument in Chapter 17 of [**16**]).

Note that the proof of Theorem 4.2 simply uses Theorem 3.1 locally in a neighborhood of the plane Σ_P. We can therefore find a local Stein neighborhood basis in $F(U)$, so the inverse image will be a local Stein neighborhood basis in U. Outside of U, we can intersect with level sets D_δ and a Stein neighborhood basis of the entire handle Σ, to obtain a global Stein neighborhood basis for $\overline{D} \cup \Sigma$.

$\square$

6. Handles of lower dimension

DEFINITION 6.1. Let $D \subset \mathbb{C}^n$ be a strictly pseudoconvex set with a real analytic defining function $\rho(z)$. Let $D_\epsilon := \{z \in \mathbb{C}^n \ : \ \rho(z) < \epsilon\}$. A totally real hypersurface Σ is a *handle of dimension* ℓ for D if there exists an $\epsilon_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0)$, $S_\epsilon := \Sigma \cap \partial D_\epsilon$ is isomorphic to the ℓ-sphere S^ℓ for some $\ell \leq n$, and for points P along S_ϵ, $d\rho \in T(\Sigma)$ and $T(\Sigma)/d\rho \subset T_{\mathbb{C}}(\partial D_\epsilon)$.

The main result of this section is that Theorem 5.1 still holds in the case that the handle Σ is lower dimension than the maximal n. In order to prove this, we need to verify that the results of Sections 2 through 3 are true for lower-dimensional handles.

LEMMA 6.2. *If*

$$\Omega := \left\{ z \in \mathbb{C}^n \ : \ |x^*|^2 + |x'|^2 + c|y|^2 - 1 < 0 \right\}$$

for some $c > 0$ *and*

$$\Sigma_P := \left\{ z \in \mathbb{C}^n \ : \ |x'|^2 + |y|^2 = 0 \right\}$$

where

$$|x^*|^2 := x_1^2 + \cdots + x_\ell^2 |x'|^2 := x_{\ell+1}^2 + \cdots + x_n^2$$

then there exists a Stein neighborhood basis for the set $\overline{\Omega} \cup \Sigma_P$

PROOF. For this proof, we will need to know that ou defining function is not just plurisubharmonic in complex tangent directions, but rather plurisubharmonic in all directions. Take the defining function $\rho(z)$ for an element of our Stein neighborhood basis for the ball and the n-dimensional totally real plane, where in the appropriate region we have that

$$\rho(z) = |y|^2 - \chi_\eta \left(|x|^2 \right).$$

Then for some sufficiently large constant A, we have that $\rho + A\rho^2$ is strictly plurisubharmonic in a neighborhood of the boundary. There exists a new function

$$\tilde{\chi} : \mathbb{R} \to \mathbb{R}$$

such that

$$\rho(z) + A\rho(z)^2 = |y|^2 - \tilde{\chi} \left(|x|^2 \right).$$

Replace ρ by $\rho + A\rho^2$, and χ_η by $\tilde{\chi}$.

Given a small $\epsilon > 0$, we follow the form of equation (2.16) to define a cylindrically symmetrical neighborhood of $\overline{\Omega} \cup \Sigma_P$:

$$\rho(z) = \begin{cases} |y|^2 + |x|^2 - (1 + 2\epsilon) & \text{if} \qquad |x^*|^2 \leq a_1 \\ |x'|^2 + |y|^2 - \chi_\eta \left(|x^*|^2 \right) + \delta & \text{if } a_1 \leq |x^*|^2 \leq a_4 \\ |x'|^2 + |y|^2 - \delta_1 & \text{if } a_4 \leq |x^*|^2 \end{cases} \tag{6.1}$$

with the same constants η, δ and a_j as in equation (2.16). Note that

$$\mathcal{L}_\rho \left(z, \lambda \right) = \begin{cases} 1 & \text{if} \qquad |x^*|^2 < a_1 \\ \frac{1}{2} + \frac{1}{2} \left(|\lambda_{n-\ell}|^2 + \cdots + |\lambda_n|^2 \right) & \text{if } a_4 < |x^*|^2 \end{cases}$$

so we only need to verify that the set $\{z \ : \ \rho(z) < 0\}$ is strictly pseudoconvex in the region $a_1 < |x^*|^2 < a_4$. Writing $|y|^2 = |y^*|^2 + |y'|^2$, we see that when $\ell < n$, ρ becomes

$$\rho(z) = |x'|^2 + |y'|^2 + \underbrace{|y^*|^2 - \chi_\eta\left(|x^*|^2\right)}_{\rho^*(z)}.$$

so that for a vector $\lambda = \lambda^* + \lambda'$, the Levi form is

$$\mathcal{L}_\rho\,(z, \lambda) = |\lambda'|^2 + \mathcal{L}_{\rho^*}\,(z, \lambda*)$$

each term of which is larger than 0 by assumption. □

LEMMA 6.3. *If*

$$\Omega := \left\{z \in \mathbb{C}^n \ : \ |x^*|^2 + |x'|^2 - c|y|^2 - 1 < 0\right\}$$

for some $0 \leq c < 1$ *and*

$$\Sigma_P := \left\{z \in \mathbb{C}^n \ : \ |x'|^2 + |y|^2 = 0\right\}$$

where

$$|x^*|^2 := x_1^2 + \cdots + x_\ell^2$$

and

$$|x'|^2 := x_{\ell+1}^2 + \cdots + x_n^2$$

then there exists a Stein neighborhood basis for the set $\overline{\Omega} \cup \Sigma_P$

PROOF. The proof of this lemma is a sraightforward adaptation of the proof of Theorem 3.1. The only difference is that the set Ω_2 is now taken to be a Stein neighborhood basis element from Lemma 6.2. □

THEOREM 6.4. *Let* $\Omega \subset \mathbb{C}^n$ *be a pseudoconvex domain with real analytic boundary* $\partial\Omega$*, and let* Σ *be a handle of dimension* ℓ *for* Ω*. Then there exists a Stein neighborhood basis for the set* $\overline{\Omega} \cup \Sigma$.

PROOF. The proof of this theorem also is a sraightforward adaptation. The proof of Theorem 5.1 may be taken with the substitution of using Lemma 6.3 in place of Theorem 3.1. □

APPENDIX A

Detailed Levi Form Calculations

We want to obtain the Levi form of a function $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ of the form $\phi(z) = (|y|^2 - \chi(|x|^2))$ in terms of derivatives of χ. To do so, we compute $\frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}$ in terms of derivatives of χ. Since $|x|^2 = x_1^2 + x_2^2$ and $|y|^2 = y_1^2 + y_2^2$, the chain rule yields

$$\frac{\partial \phi}{\partial z_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right)\left(|y|^2 - \chi\left(|x|^2\right)\right) = -x_1\chi'\left(|x|^2\right) - iy_1$$

so that

$$\frac{\partial^2 \phi}{\partial z_1 \partial \overline{z}_1} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right)\left(-x_1\chi'\left(|x|^2\right) - iy_1\right) = \frac{1}{2} - \frac{\chi'}{2} - x_1^2\chi''$$

and

$$\frac{\partial^2 \phi}{\partial z_1 \partial \overline{z}_2} = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right)\left(-x_1\chi'\left(|x|^2\right) - iy_1\right) = -x_1x_2\chi''.$$

Similarly, we obtain

$$\begin{aligned}\frac{\partial^2 \phi}{\partial z_2 \partial \overline{z}_1} &= -x_2x_1\chi'' \\ \frac{\partial^2 \phi}{\partial z_2 \partial \overline{z}_2} &= \frac{1}{2} - \frac{\chi'}{2} - x_2^2\chi''\end{aligned}$$

so that the Levi form for ϕ at any complex tangent vector λ to $\partial\Omega$ is

$$\begin{aligned}\mathcal{L}(\phi, \lambda) = \frac{1}{2}|\lambda|^2 - \Bigg[\left(\frac{\chi'}{2} + x_1^2\chi''\right)\lambda_1\overline{\lambda_1} + x_1x_2\chi''\lambda_1\overline{\lambda_2}& \\ +x_2x_1\chi''\lambda_2\overline{\lambda_1} + \left(\frac{\chi'}{2} + x_2^2\chi''\right)\lambda_2\overline{\lambda_2}\Bigg]&\end{aligned}$$

Now a complex tangent vector λ is defined by $\sum \frac{\partial \phi}{\partial z_i}\lambda_i = 0$, so in our case where the dimension is 2, we may take

$$\begin{aligned}\lambda &= \left(\frac{\partial \phi}{\partial z_2}, -\frac{\partial \phi}{\partial z_1}\right) \\ &= \left(-iy_2 - x_2\chi', iy_1 + x_1\chi'\right)\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{L}(\phi,\lambda) = {} & \frac{1}{2}\left(y_1^2 + y_2^2 + (x_1^2 + x_2^2)\left(\chi'\right)^2\right) - \left(\frac{\chi'}{2} + x_1^2\chi''\right)\left(y_2^2 + x_2^2\left(\chi'\right)^2\right) \\
& - x_1 x_2 \chi''\left(-y_1 y_2 - x_1 x_2\left(\chi'\right)^2 - i(x_1 y_2 - x_2 y_1)\chi'\right) \\
& - x_1 x_2 \chi''\left(-y_1 y_2 - x_1 x_2\left(\chi'\right)^2 + i(x_1 y_2 - x_2 y_1)\chi'\right) \\
& - \left(\frac{\chi'}{2} + x_2^2\chi''\right)\left(y_1^2 + x_1^2\left(\chi'\right)^2\right) \\
= {} & \frac{1}{2}\left(|y|^2 + |x|^2\left(\chi'\right)^2\right) - |y|^2\frac{\chi'}{2} - |x|^2\frac{\left(\chi'\right)^3}{2} - \left(x_1^2 y_2^2 + x_2^2 y_1^2\right)\chi'' \\
& - 2x_1^2 x_2^2\left(\chi'\right)^2\chi'' + 2x_1 x_2 y_1 y_2 \chi'' \\
& + 2x_1^2 x_2^2\left(\chi'\right)^2\chi'' \\
= {} & \frac{1}{2}\left(|y|^2 + |x|^2\left(\chi'\right)^2\right) - \frac{1}{2}\chi'\left(|y|^2 + |x|^2\left(\chi'\right)^2\right) \\
& - (x_1 y_2 - x_2 y_1)^2\chi'' \\
= {} & \frac{1}{2}\left(|y|^2 + |x|^2\left(\chi'\right)^2\right)(1 - \chi') - (x_1 y_2 - x_2 y_1)^2\chi''.
\end{aligned}$$

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